Transport Cost and Optimal Number of Public Facilities

Kazuo Yamaguchi*

Graduate School of Economics, University of Tokyo

June 14, 2006

Abstract

We consider the number and location problem of public facilities without obligation of use. We characterize the efficient provision condition and the relationship between transport cost and the optimal number of facilities.

Keywords: Public facilities without obligation of use; Transport cost; Congestion

JEL Classification: R53; H41

*Mail address: 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-8654, Japan; Tel.: +81-3-3944-1356; E-mail address: ee47003@mail.ecc.u-tokyo.ac.jp.

1 Introduction

It is common that public facilities, such as parks and libraries, are provided by centralized authorities to avoid several problems arising from their non-rivalry and non-excludability. However, in most communities, it is also common that the construction of low benefit-to-cost public facilities have been carried out usually. Recently, especially in Japan, this bad habit gradually tends to be revised upward by the trend of public opinion, and the authorities are required to analyze the cost-benefit of any project before the construction in order to allocate limited tax revenue efficiently.

Because consumers bear transport costs to travel to a public facility, and in addition, they may not enjoy by a dense of crowd, it may be less appealing or worthless for some consumers even if the facility itself is beneficial. Thus, spatial characteristics, such as infrastructure for transportation and distribution of consumers, are important in determining social value of a public facility. To accomplish provision of public facilities efficiently, the authority should deliberate how many facilities it provides and where to construct each facility at the expense of limited resources with taking these characteristics into considerations. However, the characterization of the optimal number and location of public facilities has not been investigated completely.[1]

In this paper, using a variant of the spatial differentiation model of Hotelling (1929) where transport cost of a consumer who uses a public facility depends on the distance between their locations, we consider the optimal number and location problem of public facilities without obligation of use. Especially, we investigate the effect of spatial characteristics such as infrastructure for transportation and distribution of consumers to the optimal number of public facilities.

The remainder of this paper is organized as follows: Section 2 describes the model; Section 3 characterizes the result; Section 4 concludes.

[1] A recent work by Berliant *et al.* (2006) tackled the problem by a general equilibrium welfare analysis.

2 The model

There are available homogeneous public facilities without obligation of use within a city $[0,l]$ where $l > 0$. The quantity of the public facility at each location is zero (not built) or one (built). Denote a facility located at x as facility x. A set X of public facilities, where $X \subset [0,l]$ and $|X| < +\infty$, stipulates the number and location of facilities provided. It takes social cost $C(|X|)$ to provide a set X of facilities, where $C : \mathbb{R}_+ \to \mathbb{R}$ is a C^2 function of n which satisfies $\lim_{n\to 0+} C_n(n) = 0 < C'(n), C''(n)$.

Consumers are uniformly distributed with density $f > 0$ along the city. Denote a consumer located at i as consumer i. Given a set X of facilities, the consumers use one or zero unit of the facility. Let $\sigma_i^X \in \Sigma_i^X \equiv X \cup \{\emptyset\}$ be the facility consumer i uses for any i ($\emptyset$ means that she uses nothing). Let $\sigma^X \in \Sigma^X \equiv \Pi_i \Sigma_i^X$ be a profile such that the facility consumer i uses is σ_i^X for any i. Let $m_x(\sigma^X)$ be a measure of consumers using the facility x given a profile σ^X. If a consumer uses a public facility, she derives a surplus which depends on how many consumers access it, but bears her own transport cost which depends on the distance to it: if $\sigma_i^X \in X \neq \emptyset$ given a profile σ^X, consumer i obtains $v\left(m_{\sigma_i^X}(\sigma^X)\right)$ where v is a C^2 function of m which satisfies $v'(m), v''(m) \leq 0 < v(0)$, but bears her own transport cost $t\left(i - \sigma_i^X\right)^2$ where $t > 0$. If she uses nothing, her utility is 0. To sum up, given a set X of facilities and a profile $\sigma^X \in \Sigma^X$, the utility of consumer i is given by

$$u^i\left(\sigma^X; X\right) \equiv \begin{cases} 0 & \text{if } \sigma_i^X \in \{\emptyset\} \\ v\left(m_{\sigma_i^X}\left(\sigma^X\right)\right) - t\left(i - \sigma_i^X\right)^2 & \text{if } \sigma_i^X \in X \neq \emptyset. \end{cases}$$

Social valuation of a set X of facilities with a profile σ^X is given by

$$V\left(\sigma^X; X\right) \equiv \int_0^l u^i\left(\sigma^X; X\right) f di \tag{1}$$

and social welfare of a set X of facilities with a profile σ^X is given by

$$W\left(\sigma^X; X\right) \equiv V\left(\sigma^X; X\right) - C\left(|X|\right). \tag{2}$$

In this paper, we assess social welfare of a set X of facilities at a Nash equilibrium $\hat{\sigma}^X \in \Sigma^X$ in the game given the set of facilities. The problem is now to maximize (2) with respect to $\left(\hat{\sigma}^X; X\right)$.

3 The result

To begin with, it is convenient to characterize the properties of a Nash equilibrium of the game. Consider a set $X \neq \emptyset$ of public facilities. For any $x \in X$, denote a set of consumers using facility x given a profile σ^X as $D_x\left(\sigma^X\right) \equiv \left\{i \in [0,l] \mid \sigma_i^X = x\right\}$. Denote a left-hand boundary of consumers using facility x as $\dot{d}_x\left(\sigma^X\right) \equiv \inf D_x\left(\sigma^X\right)$ and a right-hand boundary of consumers using facility x as $\ddot{d}_x\left(\sigma^X\right) \equiv \sup D_x\left(\sigma^X\right)$. Let $\left(\sigma_i^X, \hat{\sigma}_{-i}^X\right)$ be a profile such that the facility consumer i uses is σ_i^X and the facility consumer j uses is $\hat{\sigma}_j^X$ for any j except i.

Lemma 1. *Take any $X \neq \emptyset$ and any Nash equilibrium $\hat{\sigma}^X \in \Sigma^X$. Take any $i, j \in [0,l]$ with $i < j$, $\hat{\sigma}_i^X \neq \emptyset$ and $\hat{\sigma}_j^X \neq \emptyset$. Then, we have $\hat{\sigma}_i^X \leq \hat{\sigma}_j^X$.*

Proof. Suppose, on the contrary, $\hat{\sigma}_j^X < \hat{\sigma}_i^X$. Consider a deviation of consumer i from $\hat{\sigma}_i^X$ to $\hat{\sigma}_j^X$. Then, we have

$$\begin{aligned}
u^i\left(\hat{\sigma}_j^X, \hat{\sigma}_{-i}^X\right) &= v\left(m_{\hat{\sigma}_j^X}\left(\hat{\sigma}_j^X, \hat{\sigma}_{-i}^X\right)\right) - t\left(i - \hat{\sigma}_j^X\right)^2 \\
&= v\left(m_{\hat{\sigma}_j^X}\left(\hat{\sigma}_j^X, \hat{\sigma}_{-j}^X\right)\right) - t\left(i - \hat{\sigma}_j^X\right)^2 \\
&\geq v\left(m_{\hat{\sigma}_i^X}\left(\hat{\sigma}_i^X, \hat{\sigma}_{-j}^X\right)\right) - t\left(j - \hat{\sigma}_i^X\right)^2 + t\left(j - \hat{\sigma}_j^X\right)^2 - t\left(i - \hat{\sigma}_j^X\right)^2 \\
&> v\left(m_{\hat{\sigma}_i^X}\left(\hat{\sigma}_i^X, \hat{\sigma}_{-i}^X\right)\right) - t\left(i - \hat{\sigma}_i^X\right)^2 \\
&= u^i\left(\hat{\sigma}^X\right).
\end{aligned}$$

Hence, she can increase her payoff by the deviation, which is a contradiction. □

Lemma 2. *Take any $X \neq \emptyset$ and any Nash equilibrium $\hat{\sigma}^X \in \Sigma^X$. Then, $D_x\left(\hat{\sigma}^X\right)$ is a non-empty interval for any $x \in X$.*

Proof. First, we show that $D_x\left(\hat{\sigma}^X\right)$ is not empty. Suppose, on the contrary, $D_x\left(\hat{\sigma}^X\right) = \emptyset$. Consider a deviation of consumer x from $\hat{\sigma}_x^X$ to x. Then, we have

$$u^x\left(x, \hat{\sigma}_{-x}^X\right) = v\left(m_x\left(x, \hat{\sigma}_{-x}^X\right)\right) - t\left(x - x\right)^2 = v\left(0\right) > u^x\left(\hat{\sigma}^X\right).$$

Hence, she can increase her payoff by the deviation, which is a contradiction.

Next, we show that $D_x\left(\hat{\sigma}^X\right)$ is an interval. If $\dot{d}_x\left(\hat{\sigma}^X\right) = \ddot{d}_x\left(\hat{\sigma}^X\right)$, the proof ends. Suppose $\dot{d}_x\left(\hat{\sigma}^X\right) \neq \ddot{d}_x\left(\hat{\sigma}^X\right)$. Take any $i \in \left(\dot{d}_x\left(\hat{\sigma}^X\right), \ddot{d}_x\left(\hat{\sigma}^X\right)\right)$. Note that there exists $j \in \left[\dot{d}_x\left(\hat{\sigma}^X\right), i\right)$ and $k \in \left(i, \ddot{d}_x\left(\hat{\sigma}^X\right)\right]$ such that $\hat{\sigma}_j^X = \hat{\sigma}_k^X = x$. Suppose $x \leq i$. Given the other consumers' strategies, if consumer i uses facility x, her utility is

$$\begin{aligned}
u^i\left(x, \hat{\sigma}_{-i}^X\right) &= v\left(m_x\left(x, \hat{\sigma}_{-i}^X\right)\right) - t\left(i - x\right)^2 \\
&= v\left(m_x\left(x, \hat{\sigma}_{-k}^X\right)\right) - t\left(i - x\right)^2 \\
&> v\left(m_x\left(x, \hat{\sigma}_{-k}^X\right)\right) - t\left(k - x\right)^2 \\
&= u^k\left(\hat{\sigma}^X\right) \\
&\geq u^k\left(\emptyset, \hat{\sigma}_{-k}^X\right) \\
&= u^i\left(\emptyset, \hat{\sigma}_{-i}^X\right).
\end{aligned}$$

Therefore, we have $\hat{\sigma}_i^X \neq \emptyset$. In the case where $i < x$, the similar argument also holds. By Lemma 1, we conclude that $\hat{\sigma}_i^X = x$. $\square$

By Lemmas 1 and 2, for any $X = \left(x_j\right)_{j=1}^n$ where $x_1 < \cdots < x_n$ and any Nash equilibrium $\hat{\sigma}^X \in \Sigma^X$, we have

$$0 \leq \dot{d}_{x_1}\left(\hat{\sigma}^X\right) \leq \ddot{d}_{x_1}\left(\hat{\sigma}^X\right) \leq \cdots \leq \dot{d}_{x_n}\left(\hat{\sigma}^X\right) \leq \ddot{d}_{x_n}\left(\hat{\sigma}^X\right) \leq l$$

and all consumers in $\left(\dot{d}_x\left(\hat{\sigma}^X\right), \ddot{d}_x\left(\hat{\sigma}^X\right)\right)$ use facility x. In addition, social valuation (1) of a set X of public facilities with a Nash equilibrium $\hat{\sigma}^X \in \Sigma^X$ can be rewritten as

$$\begin{aligned} V\left(\hat{\sigma}^X; X\right) &= \sum_{x \in X} \int_{\dot{d}_x\left(\hat{\sigma}^X\right)}^{\ddot{d}_x\left(\hat{\sigma}^X\right)} \left\{v\left(m_x\left(\hat{\sigma}^X\right)\right) - t\left(i-x\right)^2\right\} f di \\ &= \sum_{x \in X} \left[v\left(m_x\left(\hat{\sigma}^X\right)\right) m_x\left(\hat{\sigma}^X\right) - \frac{1}{3} t f \left\{\left(\ddot{d}_x\left(\hat{\sigma}^X\right) - x\right)^3 - \left(\dot{d}_x\left(\hat{\sigma}^X\right) - x\right)^3\right\}\right] \end{aligned}$$

with $m_x\left(\hat{\sigma}^X\right) = \left(\ddot{d}_x\left(\hat{\sigma}^X\right) - \dot{d}_x\left(\hat{\sigma}^X\right)\right) f$.

In the following, we characterize the optimal number and location of public facilities. The characterization consists of two steps: first, we consider the optimal location of public facilities given the number of them provided; next, given optimal location, we determine the optimal number of facilities.

3.1 Optimal location given the number of public facilities

In this subsection, we consider the optimal location of public facilities given the number of them provided. Take any $n \in \mathbb{N}$ and consider a set $X = \left(x_j\right)_{j=1}^n$ of public facilities where $x_1 < \cdots < x_n$.

Given t, v and f, let $\epsilon > 0$ be the (unique) number which satisfies

$$v\left(2\epsilon f\right) = t\epsilon^2.$$

Note that if a length of the interval of consumers using a certain facility is greater than 2ϵ, the farthest consumer who uses the facility derives a surplus less than or equal to $v\left(2\epsilon f\right)$ but must bear transport cost greater than $t\epsilon^2$, which implies that her utility is negative. Thus, 2ϵ corresponds to a (potential) maximal length of the interval a single facility can attract.

Hereafter, we assume an additional condition on congestiability of facilities, summarized as:

Assumption 1. $\left(v\left(m\right)m\right)'|_{m=2\epsilon f} > 0$.

Note that $v\left(m\right)m$ represents consumers' surplus from a facility if a measure of the consumers using it is m and satisfies $\left(v\left(m\right)m\right)'' \leq 0$. Noting also that $2\epsilon f$ is a (potential) maximal measure of consumers using a facility, this assumption assures that congestion is not so tragic such that the pie from a facility is damaged by congestion in a Nash equilibrium.

For illustrative purposes, the following two examples, where the proofs are in Appendix, are now in order:

Example 1. *Suppose* $n < \frac{l}{2\epsilon}$. *If* $|x_1 - 0|, |l - x_n| \geq \epsilon$ *and* $|x_j - x_{j-1}| \geq 2\epsilon$ *for any* $j \in \{2, \cdots, n\}$, *for any Nash equilibrium* $\hat{\sigma}^X \in \Sigma^X$, *we have*

$$\dot{d}_{x_j}\left(\hat{\sigma}^X\right) = x_j - \epsilon$$

$$\ddot{d}_{x_j}\left(\hat{\sigma}^X\right) = x_j + \epsilon$$

for any $x_j \in X$. *Moreover, we have* $V\left(\hat{\sigma}^X; X\right) = \frac{4}{3}t\epsilon^3 fn$.

Example 2. *Suppose* $n \geq \frac{l}{2\epsilon}$. *If* $|x_1 - 0|, |l - x_n| = \frac{l}{2n}$ *and* $|x_j - x_{j-1}| = \frac{l}{n}$ *for any* $j \in \{2, \cdots, n\}$, *for any Nash equilibrium* $\hat{\sigma}^X \in \Sigma^X$, *we have*

$$\dot{d}_{x_j}\left(\hat{\sigma}^X\right) = x_j - \frac{l}{2n}$$

$$\ddot{d}_{x_j}\left(\hat{\sigma}^X\right) = x_j + \frac{l}{2n}$$

for any $x_j \in X$. *Moreover, we have* $V\left(\hat{\sigma}^X; X\right) = \left(v\left(\frac{lf}{n}\right) - \frac{tl^2}{12n^2}\right) lf$.

Our first result on the efficient provision problem of public facilities is that the locations in above examples are optimal:

Proposition 1. *Take any* $n \in \mathbb{N}$ *and consider a set* $X^* = \left(x_j^*\right)_{j=1}^n$ *of public facilities where* $x_1^* < \cdots < x_n^*$. *Then, it is optimal given the number* n *of public facilities if and only if the following conditions are met: if* $n < \frac{l}{2\epsilon}$, *we have* $|x_1^* - 0|, |l - x_n^*| \geq \epsilon$ *and* $|x_j^* - x_{j-1}^*| \geq 2\epsilon$ *for any* $j \in \{2, \cdots, n\}$; *otherwise, we have* $|x_1^* - 0|, |l - x_n^*| = \frac{l}{2n}$ *and* $|x_j^* - x_{j-1}^*| = \frac{l}{n}$ *for any* $j \in \{2, \cdots, n\}$.

Proof. Take any X with $|X| = n$ and any Nash equilibrium $\hat{\sigma}^X \in \Sigma^X$. Let $\bar{m}\left(\sigma^X\right) \equiv \frac{\sum_{x \in X} m_x\left(\sigma^X\right)}{|X|}$

and $\bar{d}_x\left(\sigma^X\right) \equiv \frac{\dot{d}_x\left(\sigma^X\right)+\ddot{d}_x\left(\sigma^X\right)}{2}$. Note that

$$V\left(\hat{\sigma}^X;X\right) \leq \sum_{x\in X}\left[v\left(m_x\left(\hat{\sigma}^X\right)\right)m_x\left(\hat{\sigma}^X\right) - \frac{1}{3}tf\left\{\left(\frac{\ddot{d}_x\left(\hat{\sigma}^X\right)-\dot{d}_x\left(\hat{\sigma}^X\right)}{2}\right)^3 - \left(-\frac{\ddot{d}_x\left(\hat{\sigma}^X\right)-\dot{d}_x\left(\hat{\sigma}^X\right)}{2}\right)^3\right\}\right]$$

$$\text{with eq. iff } x = \bar{d}_x\left(\hat{\sigma}^X\right) \text{ for any } x\in X$$

$$= \sum_{x\in X}\left(v\left(m_x\left(\hat{\sigma}^X\right)\right)m_x\left(\hat{\sigma}^X\right) - \frac{1}{12}\frac{t}{f^2}m_x\left(\hat{\sigma}^X\right)^3\right)$$

$$\leq n\left(v\left(\bar{m}\left(\hat{\sigma}^X\right)\right)\bar{m}\left(\hat{\sigma}^X\right) - \frac{1}{12}\frac{t}{f^2}\bar{m}\left(\hat{\sigma}^X\right)^3\right)$$

$$\text{with eq. iff } m_x\left(\hat{\sigma}^X\right) = \bar{m}\left(\hat{\sigma}^X\right) \text{ for any } x\in X$$

$$= n\int_{-\frac{\bar{m}\left(\hat{\sigma}^X\right)}{2f}}^{\frac{\bar{m}\left(\hat{\sigma}^X\right)}{2f}}\left(v\left(\bar{m}\left(\hat{\sigma}^X\right)\right)-ti^2\right)fdi.$$

Suppose $n < \frac{l}{2\epsilon}$. Noting that $v\left(2\epsilon f\right) = t\epsilon^2$, we have

$$n\int_{-\frac{\bar{m}\left(\hat{\sigma}^X\right)}{2f}}^{\frac{\bar{m}\left(\hat{\sigma}^X\right)}{2f}}\left(v\left(\bar{m}\left(\hat{\sigma}^X\right)\right)-ti^2\right)fdi \leq n\int_{-\epsilon}^{\epsilon}\left(v\left(2\epsilon f\right)-ti^2\right)fdi \quad \text{with eq. iff } \bar{m}\left(\hat{\sigma}^X\right) = 2\epsilon f$$

$$= \frac{4}{3}t\epsilon^3 fn.$$

Suppose $n \geq \frac{l}{2\epsilon}$. Then, noting that $v\left(\frac{lf}{n}\right) \geq t\left(\frac{l}{2n}\right)^2$, we have

$$n\int_{-\frac{\bar{m}\left(\hat{\sigma}^X\right)}{2f}}^{\frac{\bar{m}\left(\hat{\sigma}^X\right)}{2f}}\left(v\left(\bar{m}\left(\hat{\sigma}^X\right)\right)-ti^2\right)fdi \leq n\int_{-\frac{l}{2n}}^{\frac{l}{2n}}\left(v\left(\frac{lf}{n}\right)-ti^2\right)fdi \quad \text{with eq. iff } \bar{m}\left(\hat{\sigma}^X\right) = \frac{lf}{n}$$

$$= \left(v\left(\frac{lf}{n}\right) - \frac{tl^2}{12n^2}\right)lf.$$

Thus, we obtain the result. □

Remember that 2ϵ is a (potential) maximal length of the interval a single public facility can attract. Thus, $n < \frac{l}{2\epsilon}$ corresponds to the case where the number of public facilities is relatively small so that not all consumers access the public facility no matter how it is located. The lemma implies that the public facility should be disposed as follows: (i) facilities attract as many consumers as possible; (ii) each facility attracts the same number of consumers; (iii) each facility attracts consumers in the neighborhood of it.

3.2 Optimal number of public facilities

In this subsection, we consider the optimal number of public facilities given optimal location. By Proposition 1, the problem is reduced to maximizing

$$W^*(n) \equiv V^*(n) - C(n)$$

with respect to the number n of public facilities, where

$$\begin{aligned} V^*(n) &\equiv \max_{\{(\hat{\sigma}^X;X) | \hat{\sigma}^X \in \Sigma^X \text{is NE}, |X|=n\}} V\left(\hat{\sigma}^X; X\right) \\ &= \begin{cases} \frac{4}{3} t\epsilon^3 f n & \text{if } n < \frac{l}{2\epsilon} \\ \left(v\left(\frac{lf}{n}\right) - \frac{tl^2}{12n^2}\right) lf & \text{otherwise.} \end{cases} \end{aligned}$$

For convenience, we neglect the integer problem of the number of public facilities. Let $V_n^*\left(\frac{l}{2\epsilon}-\right) \equiv \lim_{n \to \frac{l}{2\epsilon}-} V_n^*(n)$ and $V_n^*\left(\frac{l}{2\epsilon}+\right) \equiv \lim_{n \to \frac{l}{2\epsilon}+} V_n^*(n)$. Note that $V_n^*\left(\frac{l}{2\epsilon}-\right) \leq V_n^*\left(\frac{l}{2\epsilon}+\right)$. Then, we have the following result:

Proposition 2. *The number n^* of public facilities is optimal only if*

$$V_n^*(n^*) = C_n(n^*)$$

holds. Moreover, if $C_n\left(\frac{l}{2\epsilon}\right) \leq V_n^*\left(\frac{l}{2\epsilon}-\right)$ or $V_n^*\left(\frac{l}{2\epsilon}+\right) \leq C_n\left(\frac{l}{2\epsilon}\right)$, the converse is also true.

There are some remarks on the proposition: (i) note first that the equation is a variant of Samuelson (1954)'s condition where the public facility should be provided up to the level at which its marginal social valuation under the optimal location is equal to its marginal cost. (ii) if $C_n\left(\frac{l}{2\epsilon}\right) \leq V_n^*\left(\frac{l}{2\epsilon}-\right)$ or $V_n^*\left(\frac{l}{2\epsilon}+\right) \leq C_n\left(\frac{l}{2\epsilon}\right)$ hold, the optimal number is uniquely determined.

(iii) even if $V_n^*\left(\frac{l}{2\epsilon}-\right) < C_n\left(\frac{l}{2\epsilon}\right) < V_n^*\left(\frac{l}{2\epsilon}+\right)$ holds, the number of the solutions to the equation is at most two so that we can easily find out the optimal number.

Given (t,v,f,l), let $n^*(t,v,f,l)$ be the optimal number of facilities. Then, for any $\theta \in \{t,f,l\}$, we have

$$\begin{aligned}\frac{\partial n^*(t,v,f,l)}{\partial \theta} &= -\frac{W_{n\theta}^*(n^*(t,v,f,l))}{W_{nn}^*(n^*(t,v,f,l))} \\ &= -\frac{V_{n\theta}^*(n^*(t,v,f,l))}{V_{nn}^*(n^*(t,v,f,l)) - C_{nn}^*(n^*(t,v,f,l))}\end{aligned}$$

everywhere except at $n^*(t,v,f,l) = \frac{l}{2\epsilon}$. In addition, if a total measure of consumers keeps constant, say, $lf = F$, the effect of the city length l to the number is

$$\frac{\partial n^*\left(t,v,\frac{F}{l},l\right)}{\partial l} = -\frac{V_{nf}^*\left(n^*\left(t,v,\frac{F}{l},l\right)\right)\left(-\frac{F}{l^2}\right) + V_{nl}^*\left(n^*\left(t,v,\frac{F}{l},l\right)\right)}{V_{nn}^*(n^*(t,v,f,l)) - C_{nn}^*(n^*(t,v,f,l))}$$

everywhere except at $n^*\left(t,v,\frac{F}{l},l\right) = \frac{l}{2\epsilon}$.

The result of comparative statics is summarized in Table 1. Noting that the number $\frac{l}{2\epsilon}$ of public facilities is the boundary of whether all consumers access the public facility under the optimal location or not, $C_n\left(\frac{l}{2\epsilon}\right) < V_n^*\left(\frac{l}{2\epsilon}-\right)$ corresponds to the case where t and l are relatively low and public facilities should be provided sufficiently to attract all consumers. Instead, $V_n^*\left(\frac{l}{2\epsilon}+\right) < C_n\left(\frac{l}{2\epsilon}\right)$ corresponds to the case where t and l are relatively high and public facilities should be provided sufficiently to attract all consumers. The result of t, and l with lf constant, is summarized in next corollary:

Corollary 1. *As for the optimal number of public facilities, the followings are true:*

Table 1: The result of comparative statics

	n_t^*	n_f^*	n_l^*	$n_l^*\|_{lf=F}$
$C_n\left(\frac{l}{2\epsilon}\right) < V_n^*\left(\frac{l}{2\epsilon}-\right)$ (t, l: small)	$+$	$+$	$+$	$+$
$V_n^*\left(\frac{l}{2\epsilon}-\right) < C_n\left(\frac{l}{2\epsilon}\right) < V_n^*\left(\frac{l}{2\epsilon}+\right)$	$+$ or $-$	$+$	$+$ or 0	$+$ or $-$
$V_n^*\left(\frac{l}{2\epsilon}+\right) < C_n\left(\frac{l}{2\epsilon}\right)$ (t, l: big)	$-$	$+$	0	$-$

(i) *the effect of t to the optimal number of public facilities is positive if t is sufficiently small, and the effect is negative if t is sufficiently large;*

(ii) *if lf keeps constant, the effect of l to the optimal number of public facilities is positive if l is sufficiently small, and the effect is negative if l is sufficiently large.*

Moreover, in the case of public facilities without congestion,

(iii) *as t increases, the optimal number of public facilities increases at first but finally decreases;*

(iv) *if lf keeps constant, as l increases, the optimal number of public facilities increases at first but finally decreases.*

Corollary 1 roughly states that the optimal number of public facilities is relatively small when t is sufficiently low or high, and when l is sufficiently low or high if lf keeps constant. Why? Intuitively, this reason is as follows. If t (l with lf constant) is sufficiently low, a few public facilities are sufficient for each consumer to use the public facility with sufficiently low transport cost. Thus, marginal social valuation by an additional public facility is low if these public facilities have been provided, because all consumers have already accessed the public facility with sufficiently low cost. If t (l with lf constant) is sufficiently high, marginal social valuation by an additional public facility is originally low, because consumers continue to bear high cost to access the public facility even if one public facility is added. In summary, if t (l with lf constant) is sufficiently low or high, marginal social valuation by an additional public facility is low, which implies that the optimal number of public facilities is relatively small.

4 Conclusion

In this paper, we consider the optimal number and location problem of public facilities without obligation of use. There remains several problems: (i) in this paper, we confine our attention only to an uniform distribution of consumers to simplify the analysis. However, of course, their distribution is not uniform in actuality so that an analysis with more general distribution of consumers is required; (ii) in long run, locations of public facilities are considered to be related to the land market (see Fujita (1986) and Sakashita (1987)), and the number is also thought to be related. Thus, an analysis which include the influence of land market is also important.

Appendix

In this appendix, we prove the statements in Examples 1 and 2. To prove them, the following properties, which are intuitively clear, are useful:

Lemma 3. *Take any* $X = (x_j)_{j=1}^{n}$ *where* $x_1 < \cdots < x_n$ *and any Nash equilibrium* $\hat{\sigma}^X \in \Sigma^X$. *Then, the followings are true:*

(i) $u^{\dot{d}_x(\hat{\sigma}^X)}\left(x, \hat{\sigma}^X_{-\dot{d}_x(\hat{\sigma}^X)}\right), u^{\ddot{d}_x(\hat{\sigma}^X)}\left(x, \hat{\sigma}^X_{-\ddot{d}_x(\hat{\sigma}^X)}\right) \geq 0$;

(ii) $\dot{d}_x\left(\hat{\sigma}^X\right) < \ddot{d}_x\left(\hat{\sigma}^X\right)$;

(iii) $u^{\ddot{d}_{x_j}(\hat{\sigma}^X)}\left(\hat{\sigma}^X\right) = u^{\dot{d}_{x_{j+1}}(\hat{\sigma}^X)}\left(\hat{\sigma}^X\right)$ *for any* $j \in \{1, \cdots, n-1\}$;

(iv) $\left(\dot{d}_{x_1}\left(\hat{\sigma}^X\right) - 0\right) u^{\dot{d}_{x_1}(\hat{\sigma}^X)}\left(\hat{\sigma}^X\right) = 0$, $\left(l - \ddot{d}_{x_n}\left(\hat{\sigma}^X\right)\right) u^{\ddot{d}_{x_n}(\hat{\sigma}^X)}\left(\hat{\sigma}^X\right) = 0$ *and*
$\left(\dot{d}_{x_{j+1}}\left(\hat{\sigma}^X\right) - \ddot{d}_{x_j}\left(\hat{\sigma}^X\right)\right) u^{\ddot{d}_{x_j}(\hat{\sigma}^X)}\left(\hat{\sigma}^X\right) = 0$ *for any* $j \in \{1, \cdots, n-1\}$.

Proof. Omitted. □

Property (i) states that if consumers at boundaries of those using a facility use the facility, they can obtain non-negative payoff. Property (ii) states that a measure of consumers using a facility is positive. Properties (iii) and (iv) state that the utility of a consumer at a right-hand boundary of those using a facility and that of one at a left-hand boundary of those using the right next facility are zero if they are not in the same location.

Proof of Example 1

Proof. Take any Nash equilibrium $\hat{\sigma}^X \in \Sigma^X$. We first show that $\ddot{d}_{x_1}\left(\hat{\sigma}^X\right) = x_1 + \epsilon$. Suppose $x_1 + \epsilon < \ddot{d}_{x_1}\left(\hat{\sigma}^X\right)$. If consumer $x_1 - \epsilon$ uses facility x_1, her payoff is

$$u^{x_1-\epsilon}\left(x_1, \hat{\sigma}^X_{-(x_1-\epsilon)}\right) = u^{x_1+\epsilon}\left(x_1, \hat{\sigma}^X_{-(x_1+\epsilon)}\right) > u^{\ddot{d}_{x_1}(\hat{\sigma}^X)}\left(x_1, \hat{\sigma}^X_{-\ddot{d}_{x_1}(\hat{\sigma}^X)}\right) \geq 0,$$

which implies that $\dot{d}_{x_1}\left(\hat{\sigma}^X\right) \leq x_1 - \epsilon$ and $m_{x_1}\left(\hat{\sigma}^X\right) > 2\epsilon f$. However, if consumer $\ddot{d}_{x_1}\left(\hat{\sigma}^X\right)$ uses facility x_1, her payoff is

$$u^{\ddot{d}_{x_1}\left(\hat{\sigma}^X\right)}\left(x_1, \hat{\sigma}^X_{-\ddot{d}_{x_1}\left(\hat{\sigma}^X\right)}\right) = v\left(m_{x_1}\left(x_1, \hat{\sigma}^X_{-\ddot{d}_{x_1}\left(\hat{\sigma}^X\right)}\right)\right) - t\left(\ddot{d}_{x_1}\left(\hat{\sigma}^X\right) - x_1\right)^2 < v\left(2f\epsilon\right) - t\epsilon^2 = 0,$$

which is a contradiction.

Suppose $\ddot{d}_{x_1}\left(\hat{\sigma}^X\right) < x_1 + \epsilon$. If $x_1 - \epsilon < \dot{d}_{x_1}\left(\hat{\sigma}^X\right)$, we have $m_{x_1}\left(\hat{\sigma}^X\right) < 2\epsilon f$. Consider a deviation of consumer $j \in \left(x_1 - \epsilon, \dot{d}_{x_1}\left(\hat{\sigma}^X\right)\right)$ from $\hat{\sigma}^X_j = \emptyset$ to x_1. Then, we have

$$u^j\left(x_1, \hat{\sigma}^X_{-j}\right) = v\left(m_{x_1}\left(x_1, \hat{\sigma}^X_{-j}\right)\right) - t\left(j - x_1\right)^2 > v\left(2f\epsilon\right) - t\epsilon^2 = 0 = u^j\left(\emptyset, \hat{\sigma}^X_{-j}\right) = u^j\left(\hat{\sigma}^X\right),$$

which is a contradiction. Thus, we must have $\dot{d}_{x_1}\left(\hat{\sigma}^X\right) \leq x_1 - \epsilon$. Then, we have

$$u^{\ddot{d}_{x_1}\left(\hat{\sigma}^X\right)}\left(\hat{\sigma}^X\right) \geq u^{\ddot{d}_{x_1}\left(\hat{\sigma}^X\right)}\left(x_1, \hat{\sigma}^X_{-\ddot{d}_{x_1}\left(\hat{\sigma}^X\right)}\right) > u^{\dot{d}_{x_1}\left(\hat{\sigma}^X\right)}\left(x_1, \hat{\sigma}^X_{-\dot{d}_{x_1}\left(\hat{\sigma}^X\right)}\right) \geq 0.$$

Therefore, $\dot{d}_{x_2}\left(\hat{\sigma}^X\right) = \ddot{d}_{x_1}\left(\hat{\sigma}^X\right) < x_1 + \epsilon \leq x_2 - \epsilon$. Moreover,

$$0 \leq u^{\dot{d}_{x_2}\left(\hat{\sigma}^X\right)}\left(x_2, \hat{\sigma}^X_{-\dot{d}_{x_2}\left(\hat{\sigma}^X\right)}\right) = v\left(m_{x_2}\left(x_2, \hat{\sigma}^X_{-\dot{d}_{x_2}\left(\hat{\sigma}^X\right)}\right)\right) - t\left(\dot{d}_{x_2}\left(\hat{\sigma}^X\right) - x_2\right)^2 < v\left(m_{x_2}\left(\hat{\sigma}^X\right)\right) - t\epsilon^2,$$

which implies that $m_{x_2}\left(\hat{\sigma}^X\right) < 2f\epsilon$. Therefore, $\ddot{d}_{x_2}\left(\hat{\sigma}^X\right) < x_1 + 3\epsilon \leq x_2 + \epsilon$. Then, we have

$$u^{\ddot{d}_{x_2}\left(\hat{\sigma}^X\right)}\left(\hat{\sigma}^X\right) \geq u^{\ddot{d}_{x_2}\left(\hat{\sigma}^X\right)}\left(x_2, \hat{\sigma}^X_{-\ddot{d}_{x_2}\left(\hat{\sigma}^X\right)}\right) > u^{\dot{d}_{x_2}\left(\hat{\sigma}^X\right)}\left(x_2, \hat{\sigma}^X_{-\dot{d}_{x_2}\left(\hat{\sigma}^X\right)}\right) \geq 0.$$

Therefore, $\dot{d}_{x_3}\left(\hat{\sigma}^X\right) = \ddot{d}_{x_2}\left(\hat{\sigma}^X\right) < x_2 + \epsilon \leq x_3 - \epsilon$. Moreover,

$$0 \leq u^{\dot{d}_{x_3}\left(\hat{\sigma}^X\right)}\left(x_3, \hat{\sigma}^X_{-\dot{d}_{x_3}\left(\hat{\sigma}^X\right)}\right) = v\left(m_{x_3}\left(x_3, \hat{\sigma}^X_{-\dot{d}_{x_3}\left(\hat{\sigma}^X\right)}\right)\right) - t\left(\dot{d}_{x_3}\left(\hat{\sigma}^X\right) - x_3\right)^2 < v\left(m_{x_3}\left(\hat{\sigma}^X\right)\right) - t\epsilon^2,$$

which implies that $m_{x_3}\left(\hat{\sigma}^X\right) < 2f\epsilon$. Therefore, $\ddot{d}_{x_3}\left(\hat{\sigma}^X\right) < x_2 + 3\epsilon \leq x_3 + \epsilon$. Iterating similar argument yields $\dot{d}_{x_n}\left(\hat{\sigma}^X\right) < x_n - \epsilon$ and $\ddot{d}_{x_n}\left(\hat{\sigma}^X\right) < x_n + \epsilon$. However, consider a deviation of consumer $x_n + \epsilon$ from $\hat{\sigma}^X_{x_n+\epsilon} = \emptyset$ to x_n. Then, we have

$$u^{x_n+\epsilon}\left(x_n, \hat{\sigma}^X_{-(x_n+\epsilon)}\right) > u^{\dot{d}_{x_n}(\hat{\sigma}^X)}\left(x_n, \hat{\sigma}^X_{-\dot{d}_{x_n}(\hat{\sigma}^X)}\right) \geq 0 = u^{x_n+\epsilon}\left(\emptyset, \hat{\sigma}^X_{-(x_n+\epsilon)}\right) = u^{x_n+\epsilon}\left(\hat{\sigma}^X\right),$$

which is a contradiction. Thus, we conclude that $\ddot{d}_{x_1}\left(\hat{\sigma}^X\right) = x_1 + \epsilon$.

Next, we show that $\dot{d}_{x_1}\left(\hat{\sigma}^X\right) = x_1 - \epsilon$. Suppose $\dot{d}_{x_1}\left(\hat{\sigma}^X\right) > x_1 - \epsilon$. Then, we have $m_{x_1}\left(\hat{\sigma}^X\right) < 2f\epsilon$. Consider a deviation of consumer $j \in \left(x_1 - \epsilon, \dot{d}_{x_1}\left(\hat{\sigma}^X\right)\right)$ from $\hat{\sigma}^X_j = \emptyset$ to x_1. Then, we have,

$$u^j\left(x_1, \hat{\sigma}^X_{-j}\right) = v\left(m_{x_1}\left(x_1, \hat{\sigma}^X_{-j}\right)\right) - t\left(j - x_1\right)^2 > v\left(2f\epsilon\right) - t\epsilon^2 = 0 = u^j\left(\emptyset, \hat{\sigma}^X_{-j}\right) = u^j\left(\hat{\sigma}^X\right),$$

which is a contradiction.

Suppose $\dot{d}_{x_1}\left(\hat{\sigma}^X\right) < x_1 - \epsilon$. Then, we have $m_{x_1}\left(\hat{\sigma}^X\right) > 2f\epsilon$. Consider a deviation of consumer $j \in \left(\dot{d}_{x_1}\left(\hat{\sigma}^X\right), x_1 - \epsilon\right)$ from $\hat{\sigma}^X_j = x_1$ to $\emptyset$. Then, we have

$$u^j\left(\emptyset, \hat{\sigma}^X_{-j}\right) = 0 = v\left(2f\epsilon\right) - t\epsilon^2 > v\left(m_{x_1}\left(x_1, \hat{\sigma}^X_{-j}\right)\right) - t\left(j - x_1\right)^2 = u^j\left(x_1, \hat{\sigma}^X_{-j}\right) = u^j\left(\hat{\sigma}^X\right),$$

which is a contradiction. Thus, we conclude that $\dot{d}_{x_1}\left(\hat{\sigma}^X\right) = x_1 - \epsilon$.

Iterating similar argument for $x_2, \cdots, x_n$, we obtain the result. □

Proof of Example 2

Proof. Take any Nash equilibrium $\hat{\sigma}^X \in \Sigma^X$. We first show that $\ddot{d}_{x_1}\left(\hat{\sigma}^X\right) = x_1 + \frac{l}{2n}$. Suppose $x_1 + \frac{l}{2n} < \ddot{d}_{x_1}\left(\hat{\sigma}^X\right)$. If consumer $x_1 - \frac{l}{2n}$ uses facility x_1, her utility is

$$u^{x_1 - \frac{l}{2n}}\left(x_1, \hat{\sigma}^X_{-\left(x_1 - \frac{l}{2n}\right)}\right) > u^{\ddot{d}_{x_1}(\hat{\sigma}^X)}\left(x_1, \hat{\sigma}^X_{-\left(\ddot{d}_{x_1}(\hat{\sigma}^X)\right)}\right) \geq 0,$$

which implies that $d_{x_1}\left(\hat{\sigma}^X\right) = x_1 - \frac{l}{2n}$. Note that

$$v\left(m_{x_1}\left(x_1, \hat{\sigma}^X_{-j}\right)\right) - t\left(j - x_1\right)^2 = u^j\left(x_1, \hat{\sigma}^X_{-j}\right) \geq u^j\left(x_2, \hat{\sigma}^X_{-j}\right) = v\left(m_{x_2}\left(x_2, \hat{\sigma}^X_{-j}\right)\right) - t\left(j - x_2\right)^2$$

for $j \in D_{x_1}\left(\hat{\sigma}^X\right) \cap \left(x_1 + \frac{l}{2n}, \ddot{d}_{x_1}\left(\hat{\sigma}^X\right)\right]$, which implies $m_{x_2}\left(\hat{\sigma}^X\right) > m_{x_1}\left(\hat{\sigma}^X\right) > \frac{lf}{n}$. Thus, $\ddot{d}_{x_2}\left(\hat{\sigma}^X\right) > x_2 + \frac{l}{2n}$. Note that

$$v\left(m_{x_2}\left(x_2, \hat{\sigma}_{-j}^X\right)\right) - t\left(j - x_2\right)^2 = u^j\left(x_2, \hat{\sigma}_{-j}^X\right) \geq u^j\left(x_3, \hat{\sigma}_{-j}^X\right) = v\left(m_{x_3}\left(x_3, \hat{\sigma}_{-j}^X\right)\right) - t\left(j - x_3\right)^2$$

for $j \in D_{x_2}\left(\hat{\sigma}^X\right) \cap \left(x_2 + \frac{l}{2n}, \ddot{d}_{x_2}\left(\hat{\sigma}^X\right)\right]$, which implies $m_{x_3}\left(\hat{\sigma}^X\right) > m_{x_2}\left(\hat{\sigma}^X\right) > \frac{lf}{n}$. Thus, $\ddot{d}_{x_3}\left(\hat{\sigma}^X\right) > x_3 + \frac{l}{2n}$. Iterating similar argument yields $x_n + \frac{l}{n} < \ddot{d}_{x_n}\left(\hat{\sigma}^X\right)$, which is a contradiction.

Suppose $\ddot{d}_{x_1}\left(\hat{\sigma}^X\right) < x_1 + \frac{l}{2n}$. Then, we have $m_{x_1}\left(\hat{\sigma}^X\right) < \frac{lf}{n}$. If $x_1 - \frac{l}{2n} < \dot{d}_{x_1}\left(\hat{\sigma}^X\right)$, consider a deviation of consumer $j \in \left(x_1 - \frac{l}{2n}, \dot{d}_{x_1}\left(\hat{\sigma}^X\right)\right)$ from $\hat{\sigma}_j^X$ to x_1. Then, we have

$$u^j\left(x_1, \hat{\sigma}_{-j}^X\right) = v\left(m_{x_1}\left(x_1, \hat{\sigma}_{-j}^X\right)\right) - t\left(j - x_1\right)^2 > v\left(\frac{fl}{n}\right) - t\left(\frac{l}{2n}\right)^2 \geq 0 = u^j\left(\emptyset, \hat{\sigma}_{-j}^X\right) = u^j\left(\hat{\sigma}^X\right),$$

which is a contradiction. Thus, we must have $\dot{d}_{x_1}\left(\hat{\sigma}^X\right) = x_1 - \frac{l}{2n}$. Then, we have

$$u^{\ddot{d}_{x_1}\left(\hat{\sigma}^X\right)}\left(\hat{\sigma}^X\right) \geq u^{\ddot{d}_{x_1}\left(\hat{\sigma}^X\right)}\left(x_1, \hat{\sigma}_{-\ddot{d}_{x_1}\left(\hat{\sigma}^X\right)}^X\right) > u^{\dot{d}_{x_1}\left(\hat{\sigma}^X\right)}\left(x_1, \hat{\sigma}_{-\dot{d}_{x_1}\left(\hat{\sigma}^X\right)}^X\right) \geq 0.$$

Therefore, $\dot{d}_{x_2}\left(\hat{\sigma}^X\right) = \ddot{d}_{x_1}\left(\hat{\sigma}^X\right) < x_1 + \frac{l}{2n} = x_2 - \frac{l}{2n}$. Moreover,

$$v\left(m_{x_1}\left(x_1, \hat{\sigma}_{-j}^X\right)\right) - t\left(j - x_1\right)^2 = u^j\left(x_1, \hat{\sigma}_{-j}^X\right) \leq u^j\left(x_2, \hat{\sigma}_{-j}^X\right) = v\left(m_{x_2}\left(x_2, \hat{\sigma}_{-j}^X\right)\right) - t\left(j - x_2\right)^2$$

for $j \in D_{x_2}\left(\hat{\sigma}^X\right) \cap \left[\dot{d}_{x_2}\left(\hat{\sigma}^X\right), x_2 - \frac{l}{2n}\right)$, which implies that $m_{x_2}\left(\hat{\sigma}^X\right) < m_{x_1}\left(\hat{\sigma}^X\right) < \frac{lf}{n}$. Therefore, $\ddot{d}_{x_2}\left(\hat{\sigma}^X\right) < x_1 + \frac{3l}{2n} = x_2 + \frac{l}{2n}$. Then, we have

$$u^{\ddot{d}_{x_2}\left(\hat{\sigma}^X\right)}\left(\hat{\sigma}^X\right) \geq u^{\ddot{d}_{x_2}\left(\hat{\sigma}^X\right)}\left(x_2, \hat{\sigma}_{-\ddot{d}_{x_2}\left(\hat{\sigma}^X\right)}^X\right) > u^{\dot{d}_{x_2}\left(\hat{\sigma}^X\right)}\left(x_2, \hat{\sigma}_{-\dot{d}_{x_2}\left(\hat{\sigma}^X\right)}^X\right) \geq 0.$$

Therefore, $\dot{d}_{x_3}\left(\hat{\sigma}^X\right) = \ddot{d}_{x_2}\left(\hat{\sigma}^X\right) < x_2 + \frac{l}{2n} = x_3 - \frac{l}{2n}$. Moreover,

$$v\left(m_{x_2}\left(x_2, \hat{\sigma}^X_{-j}\right)\right) - t\left(j - x_2\right)^2 = u^j\left(x_2, \hat{\sigma}^X_{-j}\right) \leq u^j\left(x_3, \hat{\sigma}^X_{-j}\right) = v\left(m_{x_3}\left(x_3, \hat{\sigma}^X_{-j}\right)\right) - t\left(j - x_3\right)^2$$

for $j \in D_{x_3}\left(\hat{\sigma}^X\right) \cap \left[\dot{d}_{x_3}\left(\hat{\sigma}^X\right), x_3 - \frac{l}{2n}\right)$, which implies that $m_{x_3}\left(\hat{\sigma}^X\right) < m_{x_2}\left(\hat{\sigma}^X\right)$. Therefore, $\ddot{d}_{x_3}\left(\hat{\sigma}^X\right) < x_2 + \frac{3l}{2n} = x_3 + \frac{l}{2n}$. Iterating similar argument yields $\dot{d}_{x_n}\left(\hat{\sigma}^X\right) < x_n - \frac{l}{2n}$ and $\ddot{d}_{x_n}\left(\hat{\sigma}^X\right) < x_n + \frac{l}{2n}$. However, consider a deviation of consumer $x_n + \frac{l}{2n}$ from $\hat{\sigma}^X_{x_n + \frac{l}{2n}} = \emptyset$ to x_n. Then, we have

$$u^{x_n + \frac{l}{2n}}\left(x_n, \hat{\sigma}^X_{-\left(x_n + \frac{l}{2n}\right)}\right) > u^{\dot{d}_{x_n}\left(\hat{\sigma}^X\right)}\left(x_n, \hat{\sigma}^X_{-\dot{d}_{x_n}\left(\hat{\sigma}^X\right)}\right) \geq 0 = u^{x_n + \frac{l}{2n}}\left(\emptyset, \hat{\sigma}^X_{-\left(x_n + \frac{l}{2n}\right)}\right) = u^{x_n + \frac{l}{2n}}\left(\hat{\sigma}^X\right),$$

which is a contradiction. Thus, we conclude that $\ddot{d}_{x_1}\left(\hat{\sigma}^X\right) = x_1 + \frac{l}{2n}$.

Next, we show that $\dot{d}_{x_1}\left(\hat{\sigma}^X\right) = x_1 - \frac{l}{2n}$. Suppose $\dot{d}_{x_1}\left(\hat{\sigma}^X\right) > x_1 - \frac{l}{2n}$. Then, we have $m_{x_1}\left(\hat{\sigma}^X\right) < \frac{lf}{n}$. Consider a deviation of consumer $j \in \left(x_1 - \frac{l}{2n}, \dot{d}_{x_1}\left(\hat{\sigma}^X\right)\right)$ from $\hat{\sigma}^X_j = \emptyset$ to x_1. Then, we have

$$u^j\left(x_1, \hat{\sigma}^X_{-j}\right) = v\left(m_{x_1}\left(x_1, \hat{\sigma}^X_{-j}\right)\right) - t\left(j - x_1\right)^2 > v\left(\frac{lf}{n}\right) - t\left(\frac{l}{2n}\right)^2 \geq 0 = u^j\left(\emptyset, \hat{\sigma}^X_{-j}\right) = u^j\left(\hat{\sigma}^X\right),$$

which is a contradiction. Thus, we conclude that $\dot{d}_{x_1}\left(\hat{\sigma}^X\right) = x_1 - \frac{l}{2n}$.

Iterating similar argument for $x_2, \cdots, x_n$, we obtain the result. $\square$

References

[1] Berliant, M., S.K. Peng and P. Wang, 2006, "Welfare analysis of the number and locations of local public facilities," *Regional Science and Urban Economics* 36, 207-226.

[2] Fujita, M., 1986, "Optimal location of public facilities: area dominance approach," *Regional Science and Urban Economics* 16, 241-268.

[3] Hotelling, H., 1929, "Stability in competition," *Economic Journal* 39, 41-57.

[4] Sakashita, N., 1987, "Optimum location of public facilities under the influence of the land market," *Journal of Regional Science* 27, 1-12.

[5] Samuelson, P.A., 1954, "The pure theory of public expenditure," *Review of Economics and Statistics* 36, 387-389.