Freemium as Optimal Menu Pricing*

Susumu Sato†

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In online contents markets, content providers collect revenues from both consumers and advertisers by segmenting consumers who are willing to avoid advertisements and who are not. To analyze such situations, I construct a model of menu pricing by advertising platforms in two-sided markets. I find that, under a linear environment, although a monopolistic platform can choose any menu of price-advertisement pairs, the optimal menu consists of only *two* services: ad-supported basic service and ad-free premium service. This menu pricing is well known as *freemium*. Furthermore, freemium remains to be an equilibrium menu pricing even under duopoly.
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1. Introduction

Freemium is a business model which is coined as a combination of the words *free* and *premium*. This word describes "a business model in which you give a core product away for free to a large group of users and sell premium products to a smaller fraction of this user base.[1]" The purpose of this paper is to show that this business model is optimal menu pricing for advertising platforms.

There are many instances of freemium in digital economy. As in Table 1, fair amount of major music- and video-streaming services adopt freemium business models. A prominent example of freemium business is Spotify, a music-streaming service with the largest market share in the world. Spotify offers two services, Free and Premium. In Free service, customers can shuffle several given playlists, with advertising audios interrupting in the time between songs. Customers who pay \$9.99 of monthly fee to subscribe Premium service can play any songs with better sound quality, create their original playlists, download musics, and listen offline, without being interrupted by advertisements. As another

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†Graduate School of Economics, The University of Tokyo. 7-3-1, Hongo, Bunkyo-ku, Tokyo, Japan. 113-0033. e-mail: susumusato@live.jp.
[1] Freemium.org "What is Freemium?": http://www.freemium.org/what-is-freemium-2/

Service	Business Model
Spotify	Freemium
Apple Music	Subscription
Pandora	Freemium
Rhapsody	Subscription
Tidal	Subscription
Deezer	Freemium

Music-streaming services

Service	Business Model
YouTube	Freemium
Netflix	Subscription
Hulu	Subscription
Bing Videos	Ad-supported
Vimeo	Freemium
Daily Motion	Ad-supported

Video-streaming services

Table 1: Business models of major streaming services. Ad-supported business model offers free services to consumers and collect revenues from advertisers. Subscription business model offers paid services to consumers. The classification of business models is by my own.

example of freemium, YouTube, a well-known ad-supported video-streaming platform, recently started to offer a paid and ad-free membership service, called YouTube Red. YouTube Red also has several additional functionalities such as saving videos on mobile devices or viewing original contents. Users of YouTube who want to avoid advertisements or get richer functionalities can upgrade their accounts to YouTube Red.

This type of business models can be seen as a class of second-degree price discrimination since the firm offers a menu of services (free and premium) and lets customers choose between them. A distinctive feature of this business model is that it uses the amount of advertisements as an instrument to screen customers.[2] Customers can choose ad-supported free service or ad-free premium service according to their nuisance from advertisements, and advertisers can show their advertisements only to free customers. Put differently, this is a price discrimination by a two-sided platform using the levels of interactions between agents on both sides as an instrument to price-discriminate.[3] This form of price discrimination is relatively new and thus have been subject to few research until recently.

There is a tradeoff when the platform uses the levels of interactions as instruments to price-discriminate. Consumers who want to enjoy contents without being annoyed by advertisements are willing to pay more to reduce the amount of advertisements. Thus, the platform can collect revenues from consumers by introducing a service with a fewer amount of advertisements and charging a higher fee. However, while offering a service with fewer advertisement may successfully collect revenues from consumers, this reduces the revenue from advertisers as the total view of advertisements shrinks. Thus, the platform need to take this tradeoff into account when they decide how to price-discriminate.

Treating freemium as a price discrimination by two-sided platforms, several questions arise; What is the optimal price discrimination for platforms? Is the freemium optimal price discrimination? Since platforms can potentially consider any nonlinear advertisement-

[2] Of course there is another instrument of price discrimination and another form of freemium business model where free service and premium service differ in their intrinsic functionalities. Under this kind of business model, customers choose services according to their preference on the functionalities. This kind of freemium businesses include online applications (e.g., Evernote), publishers (IDES), massive open online courses (Coursera), and so on. This kind of business models can be treated as versioning as mentioned in Section 1.1

[3] For a brief review of the economics of two-sided markets, see Rysman (2009).

price path to maximize its profit, and freemium is just one special class of such a price discrimination, it is natural to think that there would be better ways to collect revenues for the platforms. On the flip side of the coin, for freemium to be optimal, this must be superior to any other candidate menus platforms can design.

To answer these questions, I construct a model of menu pricing problem of advertising platforms in two-sided markets where consumers are annoyed by advertisements and advertisers benefit from listing advertisements. The platform potentially can offer any menu of services which specify the pairs of amount of advertisements and fixed fees.

My main result (Proposition 2.1) shows that *the optimal menu pricing should be freemium* in certain cases. More precisely, I show that, under the linear specification, which is commonly adopted in the literature of advertising platforms, the monopolistic platform optimally offers only *two* services: basic service with full advertisements and premium service with no advertisements. This menu pricing segments consumers into two groups: those who contribute to the platform's revenue by paying premium fees and those who contribute by viewing advertisements, leaving no intermediate segment of consumers. In fact, this simple segmentation is optimal when consumer nuisance from advertisements is linear, and the platform successfully collect revenues from both consumers and advertisers.

Then, I analyze several properties of freemium pricing. First, I examine welfare properties of freemium pricing. I show that profit maximizing platform always underprovides advertisers and consumers who view advertisements (Proposition 2.2). As a result, the size of advertising network is too small relative to the social optimum. Next, I compare freemium with another business model called ad-supported business model. I find that, the platform provides more advertisers and less consumers who view advertisements under freemium (Proposition 2.3). This difference stems from the difference in the appropriability of surplus from consumers who avoids advertisements. Under ad-supported business model, these consumers do not participate the platform, and the platform collects no revenues . On the other hand, a platform which adopts freemium can collect revenues from these consumers by providing ad-free services. This generates the incentive toward reducing (increasing) the amount of consumers who view (do not view) advertisements. And this in turn reduces the average nuisance of consumers who view advertisements, and thus platform increases the amount of advertisers.

I also examine whether freemium remains to be valid under different situations. The property that platforms offer only two services remains to be valid under a duopoly situation (Proposition 3.1). In this sense, my main result that freemium is optimal, or an equilibrium, is robust to the competition.

In summary, the results in my paper provide economic foundations for the prevalence of freemium business models; among a lot of alternatives, offering only two free and premium services is actually the best strategy for platforms.

The rest of the paper proceeds as follows. In the next subsection, I review the related literature. Section 2 presents a model of menu pricing by a monopoly platform and main results. Section 3 presents a doupoly extension, and Section 4 concludes.

1.1. Related Literature

There are three groups of researches related to my paper: literature on price discrimination, two-sided markets, and freemium.

Price Discrimination There is a huge literature on second-degree price discrimination (e.g., Mussa and Rosen (1978), Maskin and Riley (1984)). Papers related to my model are those which focus on the optimality of price discrimination, or versioning (e.g., Salant (1989), Deneckere and McAfee (1996), Varian (1997), Jing (2007), and Anderson and Dana (2009)).

Salant (1989) examines the condition under which second-degree price discrimination is suboptimal, and Deneckere and McAfee (1996) examine the profitability of price discrimination through the introduction of “damaged goods". These analyses are further developed by Varian (1997) and Anderson and Dana (2009). A common finding in these papers is that, if consumer preference and production cost are linear in quality, inducing self-selection through price discrimination will never be optimal. Intuition behind this result is that, when consumer preference and quality cost is linear in quality, marginal profit of increasing the quality for each consumer is constant and it is optimal to increase the quality as long as possible if the marginal profit is positive and not to provide the goods if the marginal profit is negative. Then, the monopolist optimally segments consumers into those who use the service and those who are excluded from the service. This segmentation is achieved by the uniform monopoly pricing on the good with highest quality.

Contrary to these researches, I show that even under the linear environment, price-discrimination is optimal for the platform. Intuition behind this result is the following. In two-sided markets, each customer has its “consumer value" that reflects the willingness to pay for the service, and “input value" that reflects the profit from procuring the consumer to list the advertisements. To exploit these values at the same time, the platform optimally offers two services.

One research which obtains a similar result is Jing (2007). He examines a linear environment as specified in Salant (1989), with one exception that there are direct network externalities. He shows that, when there are network externalities, then it is optimal for the firm to offer two products which consist of a good with lowest possible quality and with highest possible quality. One difference between Jing (2007) and this paper is that, while he considers the menu pricing with direct network externalities in one-sided markets, I analyze the properties of optimal menu pricing inherent to the two-sidedness of markets. This difference in environments leads to the different behavior of optimal menu pricing and derives different implications.[4]

Two-sided markets and advertising platforms My model is based on the framework of two-sided markets (e.g., Rochet and Tirole (2003), Armstrong (2006), Weyl (2010)).

[4]Another technical difference is that, while Jing (2007) requires some exogenous bounds on the possible qualities to derive the qualities of two goods, these bounds are endogenously determined by the platform in two-sided markets. First, the platform cannot assign the amount of advertisements less than zero, which gives the upper bound on the “quality" in my model. Second, the platform also cannot assign the amount of advertisements more than the amount of advertisers who actually participate the platform, which gives the lower bound on the quality. Finally, the amount of advertisers who participate is determined by the platform. These factors endogenize the bound on possible qualities.

There is a burgeoning literature on nonlinear-pricing or price discrimination in two-sided markets (e.g., Bedre-Defolie and Calvano (2013), Choi et al. (2015), Jeon et al. (2016)). In the sense that a platform uses transaction as an instrument to price-discriminate, the model of Gomes and Pavan (2016) is the most close to mine. They consider a price-discrimination by a many-to-many matching platform where each agents are characterized by vertical types. They show the conditions under which the optimal mechanism will be a threshold rule and analyze the properties of optimal mechanisms. In this respect, they treat the broader range of environments than mine. On the other hand, by focusing on simpler environment, my model provides a tractable setting which enables more detailed analyses on the properties of optimal menu pricing, especially related to freemium. In addition, my result that freemium is adopted by platforms is robust to the competition, which is not easy to show in Gomes and Pavan's mechanism design framework.

There is also a literature on the behavior of advertising platforms in the framework of two-sided markets (e.g., Gabszewicz, Laussel, and Sonnac (2004), Anderson and Coate (2005), Anderson and Gabszewicz (2006), Peitz and Valletti (2008)). Anderson and Coate (2005) find that advertising platforms always underprovides advertisements in terms of social welfare when they can charge prices to consumers. My result is consistent with their result. The platform also underprovides advertisements under freemium. In addition, I show that the amount of advertiser is larger and the amount of consumers who view advertisements is smaller under freemium than under ad-supported model. In this respect, I show that freemium alleviates the underprovision of advertisements to consumers, while it exacerbates the underprovision of consumers to advertisers.

Freemium The word freemium is disseminated by Anderson (2009). In the area of management, Eisenmann et al. (2011) examined the Dropbox's business model as a case study of freemium business.

In economics, there are a few researches on freemium which focus on the role of combating piracies and exploiting network externalities (e.g., Halmenschlager and Waelbroeck (2014), Nan et al. (2016)). There is few research on freemium as a price discrimination by two-sided platforms. One exception is Zennyo (2016). Using an approach close to mine, he analyzes the behaviors of freemium pricing by duopoly advertising platforms. While his main focus is on the behavior of equilibrium pricing *given* that platforms adopt freemium, my focus is on the optimality of freemium in a broader class of selling procedures. My result shows that freemium is also optimal price discrimination and an equilibrium price discrimination. Thus, this paper contributes to the research on freemium pricing in a different way.

2. Monopoly Menu Pricing

In this section, I present a model of monopoly menu pricing by an advertising platform. I show that freemium, providing only two basic and premium services, is optimal menu pricing for the platform. Then, I examine the behavior of the freemium pricing and its welfare properties. Finally, I compare freemium with ad-supported business model.

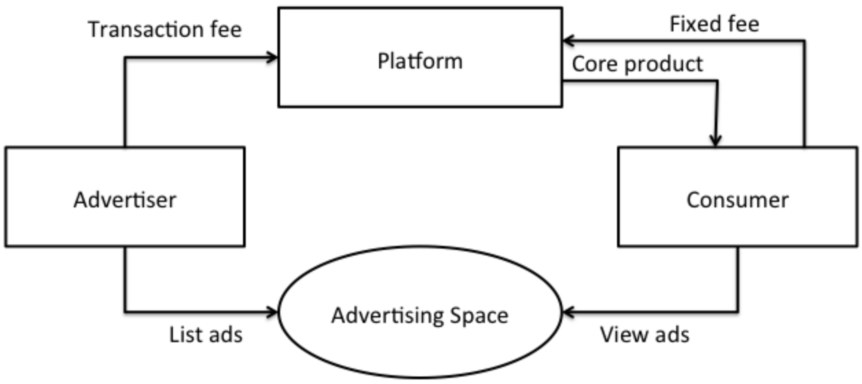

Figure 1: Flow of Transactions

2.1. Model

The model consists of three types of agents: a monopolistic *platform* which operates an advertising space, unit mass of *consumers*, and unit mass of *advertisers* who may potentially participate the platform. Main features of this model are that (i) a platform can offer a menu of services which specify the intended amount of advertisements and fixed fees, (ii) consumers derive utilities from an intrinsic value of services, but incur nuisance cost from transactions with advertisers, and (iii) advertisers benefit from transactions with consumers.

Platform The platform can offer a menu of services and an advertising space to potential consumers and advertisers. A *menu* $M \equiv (m_k)_{k=0}^{K} \in \mathbb{R}_+^{2(K+1)}$ is a profile of $(K+1)$ services. For each $k = 0, 1, \dots K$, the k-th *service* $m_k \equiv (\overline{a}_k, p_k)$ specifies a pair of an *intended amount of advertisers* $\overline{a}_k \in \mathbb{R}_+$ on that service and a *fixed fee* $p_k \in \mathbb{R}$ for using the service. The platform also charges *per-transaction fees* $p_a \in \mathbb{R}$ to advertisers. Figure 1 shows the flow of transactions.

Let $a \in \mathbb{R}_+$ be the *total amount of advertisers* who participate the platform and a_k be the *actual amount of advertisers* on k-th service. I assume that

$$a_k = \min\{\overline{a}_k, a\} \text{ for each } k. \tag{1}$$

This assumption means that the actual amount of advertisers a_k who transact with consumers on each service cannot exceed the total amount of advertisers a who participate the platform. If $\overline{a}_k \leq a$, then the platform can assign the intended amount without problems.

Let $d_k \in \mathbb{R}_+$ denote the amount of consumers who choose m_k. The amount of transaction under k-th service is given by $d_k a_k$. Under this setting, the *total amount of transaction* T is given by

$$T = \sum_{k=0}^{K} d_k a_k.$$

The platform who offers M and p_a earns revenue Π from (i) fixed fees for each services from consumers, and (ii) transaction fees from advertisers, which can be expressed as

$$\Pi = \sum_{k=0}^{K} d_k p_k + T p_a = \sum_{k=0}^{K} d_k (p_k + a_k p_a).$$

Consumers Consumers obtain an intrinsic value $v \in \mathbb{R}_+$ from participating the platform, which is independent of services chosen. For simplicity, I assume that v is the same among consumers.

I also assume that consumers are annoyed by the presence of advertisements, and these nuisance from advertisements are heterogeneous among consumers. In particular, each consumer incurs a *nuisance cost* $\tilde{c}$ per transaction with advertisers, which is privately known by the consumer and follows a strictly increasing, continuously differentiable distribution function F on $[0, C]$ with the density function f. I assume that $\frac{F(\tilde{c})}{f(\tilde{c})}$ is increasing in $\tilde{c}$. The specification that consumer nuisance cost is linear in the amount of advertisement is commonly adopted in the literature of advertising platforms (e.g., Anderson and Gabszewicz (2006)) and I follow this convention.

Imposing the quasi-linearity assumption, the utility of consumer with type $\tilde{c}$ who chooses k-th service can be expressed as $v - \tilde{c} a_k - p_k$.[5] Normalizing the value of outside option to zero, we can write the utility function of consumer with type $\tilde{c}$ as follows:

$$U(\tilde{c}) = \begin{cases} v - \tilde{c} a_k - p_k & \text{if } m_k \text{ is chosen,} \\ 0 & \text{if none is chosen.} \end{cases}$$

Finally, each consumer has a unit demand and chooses the alternative that gives the greatest utility.

Advertisers Advertisers are heterogeneous in their per-transaction benefit b, which reflects an expected profit from consumers they transact with, and is privately known by the advertiser. I assume that b follows a strictly increasing, continuously differentiable distribution function G on $[0, B]$ with the density function g, and that $\frac{1-G(b)}{g(b)}$ is decreasing in b. For simplicity, I also assume that advertisers only differ in per-transaction benefits and that there is no benefit from just participating the platform.

Each advertiser is equally assigned with the advertisement spaces, which means that each advertiser transacts with $\frac{T}{a}$ consumers on average. Thus, given the total amount of advertisers a, the total amount of transaction T, and the per-transaction fee p_a, the payoff

[5] This specification implicitly assumes that each consumer correctly forms the expectation over the realization of a_k.

of advertiser with type b is given by

$$(b - p_a)\frac{T}{a}.^6$$

We can see that an advertiser with type b participates the platform if and only if $b \geq p_q$. Thus, the demand for the advertisement space is given by

$$a = 1 - G(p_a). \tag{2}$$

Note that, since G is strictly increasing, we can invert G and write p_a as $G^{-1}(1 - a)$.

Timing Timing is as follows.

1. Platform chooses M and p_a.
2. Observing M and p_a, advertisers decide whether to participate the platform. At the same time, consumers decide which service to choose or not to participate the platform, following the correct expectation on the amount of advertisers.
3. All variables realize.

2.2. Profit Maximization

Given the setting in the previous subsection, consider the profit maximization problem of the platform.

First, note that the choice variables $\overline{a}_k$ for $k = 0, \dots, K$ and constraint (1) can be replaced by a_k for $k = 0, \dots, K$ and the constraint

$$a_k \leq a \quad \text{for each } k = 0, \dots, K, \tag{3}$$

since the platform can realize any $a_k \leq a$ by choosing the same value of $\overline{a}_k$. Hence, I consider the platform's problem as the choice of $(a_k, p_k)_{k=0}^K$ instead of $(\overline{a}_k, p_k)_{k=0}^K$.

Next, without loss of generality, assume that $a_0 \leq a_1 \leq \cdots \leq a_K$ and that $p_0 \geq p_1 \geq \cdots \geq p_K$.[7] Also, without loss of generality, assume that $p_0 \leq v$ so that m_0 has a probability to be chosen.[8] Then let c_0 be the type who is indifferent between m_0 and not buying, and let c_k be the type which is indifferent between m_k and m_{k-1} for each $k = 1, \dots, K$. To

[6] This is a natural extension of standard two-sided markets literature. For example, in the model of Rochet and Tirole (2003), the amount of transaction T between the amount d of consumers and the amount a of advertisers is given by

$$T = da.$$

In this case, the benefit function of advertisers in my model can be written as

$$(b - p_a)\frac{T}{a} = (b - p_a)d,$$

which is the same as the benefit function presented in Rochet and Tirole (2003).

[7] Further assuming that $p_0 \geq p_1 \geq \cdots \geq p_K$ is without loss of generality since if $p_k < p_{k+1}$ hold some k, then no consumer chooses m_{k+1} since m_k gives strictly greater utility for any consumer.

[8] If $p_0 > v$ and $p_1 \leq v$ we can induce the same demand by reintroducing the menu $(m'_k)_{i=0}^K$ such that $m'_k = m_{k-1}$ for $k = 1, \dots, K$ and $m'_K = m_K$.

break ties, I assume that each consumer with type c_k choose m_k rather than m_{k-1}. Then, we can see that

$$\begin{cases} c_0 = \dfrac{v - p_0}{a_0} \text{ and } c_k = \dfrac{p_{k-1} - p_k}{a_k - a_{k-1}} \text{ for } k = 1, \dots, K & \text{if } a_0 > 0 \\ c_1 = \dfrac{p_0 - p_1}{a_1} \text{ and } c_k = \dfrac{p_{k-1} - p_k}{a_k - a_{k-1}} \text{ for } k = 2, \dots, K & \text{if } a_0 = 0, \end{cases} \tag{4}$$

and that consumers with type $\tilde{c} \in [c_k, c_{k+1})$ chooses m_k. In the case where $a_0 = 0$, all types $\tilde{c} \in [0, c_1)$ will choose m_0 as long as $p_0 \leq v$. Thus, the demand for each service is given by

$$\begin{cases} d_k = F(c_k) - F(c_{k+1}) \text{ for } k = 0, \dots, K-1, \text{ and } d_K = F(c_K) & \text{if } a_0 > 0 \\ d_0 = 1 - F(c_1), d_k = F(c_k) - F(c_{k+1}) \text{ for } k = 1, \dots K-1, \text{ and } d_K = F(c_K) & \text{if } a_0 = 0. \end{cases} \tag{5}$$

Putting these elements together, the profit maximization problem of the platform can be expressed as

$$\begin{aligned} \max_{(a_k, p_k)_{k=0}^K} \quad & \sum_{k=0}^{K} d_k (p_k + a_k p_a) \\ \text{s.t.} \quad & (2), (3), (4), \text{ and } (5). \end{aligned} \tag{6}$$

Then consider the solution to the maximization problem above. I say a menu M is *freemium* if it consists of only two services. In this case, M can be expressed as $M = (m_B, m_P) \equiv ((a_B, p_B), (a_P, p_P))$ with $a_B > a_P$ and $p_B < p_P$. I call the service with lower price $m_B = (a_B, a_P)$ as *basic service* and the service with higher price $m_P = (a_P, p_P)$ as *premium service*.[9] Let c be the type of consumer who is indifferent between basic service and premium service. The following proposition sates the main result that the profit-maximizing menu should be freemium which satisfies certain properties.

Proposition 2.1. *Optimal menu is freemium, that is, the platform offers only two services at the optimum. In particular, the profit-maximizing menu is the freemium with* $((a_B, p_B), (a_P, p_P)) = ((a, p), (0, v))$*, and* c *and* p_a *are determined by the following equations:*

$$p_a = c + \frac{F(c)}{f(c)}, \tag{7}$$

$$c = p_a - \frac{1 - G(p_a)}{g(p_a)}. \tag{8}$$

Proof. In Appendix. □

The intuition behind this result is the following. When consumer nuisance costs are linear in the amount of advertisements, it is always optimal for the platform to either increase or decrease the amount of advertisements to each consumer as long as possible.

[9]The menu pricing described so far is not precisely the same as the freemium referred in the introduction, since basic goods might not be free. I use this definition of freemium because this definition induces simple derivation of optimal pricing.

Thus, for any k-th intermediate service, the amount of advertisement will be equal to one of adjacent service. As a results, only two services with full advertisements and no advertisements remains. Moreover, once we accept that the platform only offer two services, it follows that the platform equates the marginal revenue and marginal cost of increasing an agents on one side who interacts with the agents on the other side. These incentives yield the equations (7) and (8). These equations are familiar in the literature of two-sided markets which states that platforms equate the sum of transaction prices and price semi-elasticity of demand of each side (e.g., Rochet and Tirole (2003)). In other words, provided that the freemium is optimal, its behavior is fairly standard two-sided pricing.

This intuition can be stated in another way. Consider the situation where the platform chooses the amount of advertisement $a(\tilde{c})$ for each consumer with type $\tilde{c}$. We can interpret $\tilde{c} + \frac{F(\tilde{c})}{f(\tilde{c})}$ as the *virtual marginal cost* of listing an advertisement to consumer with type $\tilde{c}$, in the sense that the platform need to pay that amount to induce the consumer to view that advertisement (see Myerson (1981)). On the other hand, marginal revenue from listing an advertisement to consumer is the per-transaction fee p_a from the advertiser. When nuisance costs are linear, these marginal cost and marginal revenue are constant in $a(\tilde{c})$ and thus it is optimal to increase (decrease) $a(\tilde{c})$ as much as possible if marginal revenue p_a exceeds (falls below) the virtual marginal cost $\tilde{c} + \frac{F(\tilde{c})}{f(\tilde{c})}$. This means that the optimal amount of advertisement $a(\tilde{c})$ for the consumer with type $\tilde{c}$ greater (smaller) than threshold $\tilde{c}$ given by the equation (7) will be 0 (a). In determining the threshold type p_a of advertisers, the platform equates the marginal revenue from increasing an amount of advertisers $p_a - \frac{1-G(p_a)}{g(p_a)}$ and the cost c of keeping the consumer demand unchanged. This yields the equation (8).

I have shown that the optimal menu pricing is freemium in the sense defined above. However, this menu pricing is not precisely the same as the freemium in the real world, since the basic goods might not be free. Then we can ask when does the optimal menu pricing corresponds with the freemium in the real world. In other words, the question is when $p = 0$ holds at the optimum. The answer is that, when the benefit of advertisers from transaction is sufficiently large relative to the intrinsic value of services, then $p = 0$ is optimal for the platform.

From the equation (8), we can see that if $v \leq (1 - G(p_a))\left(p_a - \frac{1-G(p_a)}{g(p_a)}\right)$, then $p = 0$ because

$$p = v - (1 - G(p_a))\left(p_a - \frac{1 - G(p_a)}{g(p_a)}\right)$$

holds in the interior solution. That is, consumers who use the basic service need not to pay anything if the benefits of advertisers from listing advertisements are sufficiently large relative to the intrinsic value consumers derive from the platform. This is the common property which is observed in the models of two-sided markets. We can also see this inequality as the condition under which freemium in the literal sense ($p = 0$) is optimal. Then, because the platform cannot adjust the price p, its behavior slightly changes. The next result shows this property.

Result 2.1. *If $v \leq (1 - G(p_a))\left(p_a - \frac{1-G(p_a)}{g(p_a)}\right)$ holds at the optimum, then $p = 0$. In addition, c and p_a are determined by the equation*

$$\eta_c(c)(p_a - c) = p_a - \frac{1 - G(p_a)}{g(p_a)}, \tag{9}$$

where $c = \frac{v}{1-G(p_a)}$ *and* $\eta_c(c) \equiv \frac{c}{F(c)} f(c)$ *is the nuisance elasticity of demand for basic service.*

This equation can be rewritten as follows:

$$\frac{\partial c}{\partial a} \frac{1}{c} \frac{f(c)c}{F(c)} (v - ap_a) = p_a - \frac{1 - G(p_a)}{g(p_a)} \tag{10}$$

The left-hand side is the cost of increasing the amount of advertisers: the product of the percentage change in threshold type due to the increase in the amount of advertisers, surplus elasticity of demand, and the per-consumer revenue net of opportunity cost of losing premium consumers. The right-hand side is the simple marginal revenue of increasing the amount of advertisements from advertisers. These cost and benefit equate at the optimum.

This result is analogous to Gomes (2014). When the platform can use a side payment to adjust consumers' incentive, the platform just maximize the total virtual value from consumers and advertisers. On the other hand, when the platform cannot use a side payment due to the nonnegative constraint of fixed fees, the platform need to care about the demand elasticity of increasing the amount of advertisements.

Going back to the interior solution, the next result shows simple comparative statics of the behavior of freemium pricing. Roughly speaking, these results state that if either consumers' nuisance costs are more likely to be high, or advertisers' benefit is more likely to be high, then both of threshold types c and p_a will be higher.

Result 2.2. *The following comparative statics results hold.*

1. *If the distribution function F is replaced by a distribution function $\tilde{F}$ which dominates F according to reverse hazard rate,* [10] *both c and p_a increase.*

2. *Suppose the distribution function $G(b)$ has an inverse hazard rate function $\lambda(\theta, b)$ which is continuously differentiable, increasing in the first argument, and decreasing in the second argument. Then p_a and c are increasing in θ.*

Proof. In Appendix. □

The first part of comparative satatics is straightforward. The more consumer nuisance cost is likely to be high, the lower virtual marginal cost of listing an advertisement to each consumer. Then by equation (7), for a given amount of advertisements, threshold type of consumer will be higher. Next, if the threshold type of consumer will be higher, then the virtual value of marginal advertiser will be higher by equation (8). These facts lead to the increase in c and p_a.

The effects of change in the distribution G of advertisers' types are not so clear. By equation (7) we can see that threshold types p_a and c moves in the same direction regardless of the type of exogenous shock, but the direction in which these thresholds move is unclear. However, parameterizing the distribution functions G by the inverse hazard rate function $\lambda(\theta, b)$, optimal values of p_a and c turn out to be increasing in θ.

[10] A distribution function $\tilde{F}$ dominates another distribution function F according to reverse hazard rate if for any $\tilde{c} \in [0, C]$, $\frac{\tilde{f}(\tilde{c})}{\tilde{F}(\tilde{c})} \geq \frac{f(\tilde{c})}{F(\tilde{c})}$ holds.

2.3. Welfare Analysis

Consider the socially optimal menu pricing and the divergence between the social optimum and profit maximizing menu. I restrict the attention to the set of menus which contains two elements $(0, v)$ and (a, p). Actually, it can be shown that this class of menus are welfare-maximizing using the same logic as in the Proposition 2.1.

First, consider the utility of consumers. Consumer who chooses $(0, v)$ obtains 0 utility. On the other hand, the consume who chooses (a, p) obtains the utility $v - \tilde{c}a - p$. Summing these up over consumers, the consumer surplus CS is obtained as

$$\begin{aligned} CS &= \int_0^c (v - \tilde{c}a - p) f(\tilde{c}) d\tilde{c} \\ &= F(c)(1 - G(p_a))(c - \hat{c}), \end{aligned} \tag{11}$$

where $\hat{c} = E[\tilde{c}|\tilde{c} \le c]$ is the average disutility of consumers who choose the service with advertisements. On the other hand, the advertiser surplus AS is given by

$$\begin{aligned} AS &= F(c) \int_{p_a}^B (b - p_a) g(b) db \\ &= F(c)(1 - G(p_a))(\hat{b} - p_a), \end{aligned} \tag{12}$$

where $\hat{b} = E[b|b \ge p_a]$ is the average benefit of advertisers who participate the platform. Summing these and the platform's profit up, the total surplus TS is given by

$$\begin{aligned} TS = CS + AS + \Pi &= v + \int_0^c \left(\int_{p_a}^B bg(b)db - \tilde{c}a \right) f(\tilde{c}) d\tilde{c} \\ &= v + F(c)(1 - G(p_a))(\hat{b} - \hat{c}). \end{aligned} \tag{13}$$

We can see that the total surplus depends only on c and p_a. Thus, it suffices to consider the socially optimal values of c and p_a.[11] Taking derivatives with respect to c and p_a, we can obtain the welfare-maximizing pricing, which is determined by the following equations:

$$c = \hat{b} \text{ and } p_a = \hat{c}. \tag{14}$$

This means that the threshold type of one side equals the average type of the other side who interacts. The next proposition is a natural consequence of Spence (1975) and Weyl (2010); Besides the market power, the profit-maximizing behaviors deviate from the social optimum to the extent that their effects on the marginal agent and average agent diverge. Following Weyl (2010), I call this divergence as *Spence distortion*.

Proposition 2.2. *Under the profit-maximizing pricing, the following equations for threshold types c and p_a hold:*

$$c = \underbrace{\hat{b}}_{\textit{Social optimum}} + \underbrace{(p_a - \hat{b})}_{\textit{Spence distortion}} - \underbrace{\frac{F(c)}{f(c)}}_{\textit{market power distortion}} \tag{15}$$

$$p_a = \underbrace{\hat{c}}_{\textit{Social optimum}} + \underbrace{(c - \hat{c})}_{\textit{Spence distortion}} + \underbrace{\frac{1 - G(p_a)}{g(p_a)}}_{\textit{market power distortion}} \tag{16}$$

[11] Actually, any c can be chosen by the platform by choosing an appropriate value of p.

Profit-maximizing price for advertisers is too high and the amount of consumers who view advertisements is too small in terms of social welfare. In total, the amount of transaction is insufficient.

In other words, profit-maximizing size of network is too small in terms of social welfare, because both the amount of consumers who choose the service with advertisement and the amount of advertisers who participate the platform is too small. In relation to the literature, I confirm that the result of Anderson and Coate (2005) that profit-maximizing amount of advertisements is too small in terms of social welfare remains to hold even under freemium.

2.4. Comparison between Business Models

It is interesting to compare properties of freemium with those of ad-supported business model since these business models are both prevalent in real world and their difference in revenue structures might lead to different behaviors. I say the menu is *ad-supported* if it consists of only one service (a, p). Then, the pricing problem of the platform which adopts ad-supported menu is given by

$$\max_{a,p} \quad F\left(\frac{v-p}{a}\right)(p + aG^{-1}(1-a)). \tag{17}$$

Deriving the first-order conditions, we can see the following result.

Proposition 2.3. *Under the ad-supported business model, the optimal prices for consumers and advertisers are determined by the equations*

$$p_a = c + \frac{F(c)}{f(c)} - \frac{v}{a}, \tag{18}$$

$$c = p_a - \frac{1-G(p_a)}{g(p_a)}. \tag{19}$$

In addition, p_a, and c are higher than under freemium.

Proof. In Appendix. □

This result implies that the amount of advertisers is lower and the amount of consumers who view advertisements are higher under ad-supported business than under freemium. Together with the Proposition 2.3., we can see that freemium alleviates the incentive of platforms to under-provide advertisements relative to ad-supported business but exacerbates the incentive to under-provide consumers who view advertisements. This result comes from the fact that the platform cannot collect revenues from consumers who do not want to view advertisements under the ad-supported business model. Platforms which adopt freemium can collect revenues from consumers who avoid advertisements. This changes incentives of platforms in the way that more consumers pay to avoid advertisements, exacerbating the underprovision of consumers who view advertisements. On the other hand, this underprovision of consumers decreases the type c of threshold consumer, which implies that the type p_a of threshold advertiser also decreases. This means that the amount of advertisements will be higher, alleviating the underprovision of advertisements.

2.5. Heterogeneous Intrinsic Value

It seems restrictive that consumer intrinsic value from participating the platform is constant at v. Nevertheless, we can see that the qualitative result will not change even if this assumption is relaxed.

Suppose that v follows a strictly increasing, continuously differentiable distribution function H on $[0, V]$ with density h, and $\frac{1-H(v)}{h(v)}$ being decreasing in v. Setting the profit-maximization problem and deriving the first-order conditions for p_k, $k = 1, \ldots, K-1$ accordingly, we can obtain

$$\begin{aligned}&\int_{c_{k+1}}^{c_k} (1 - H(ca_k + p_k))f(\tilde{c})d\tilde{c} - \int_{c_{k+1}}^{c_k} h(\tilde{c}a_k + p_k)f(\tilde{c})dc(p_k + a_k G^{-1}(1-a)) \\ &\quad + (1 - H(c_k a_k + p_k))f(c_k)(c_k - G^{-1}(1-a)) \\ &\quad - (1 - H(c_{k+1}a_k + p_k))f(c_{k+1})(c_{k+1} - G^{-1}(1-a)) = 0.\end{aligned} \tag{20}$$

This has a solution $c_k = c_{k+1}$. Thus, the same result is observed.

3. Duopoly Menu Competition

In this section, I extend the model to duopoly competition. The qualitative result that the equilibrium menu pricing or optimal monopolistic menu pricing should be freemium remains to be valid.

Consider a duopoly case of the previous section. I adopt a Hotelling specification. There are two firms located on the edges of a unit interval, and consumers are uniformly distributed on that interval. Let $M_i = (m_{ki})_{k=0}^K$ be menu that firm i offers. I assume that advertisers multihome so that there is no competition between platforms for advertisers. Consumer's utility from participating the platform i's service is given by

$$U_i(\tilde{c}, x) = \begin{cases} v - \tilde{c}a_{ki} - p_{ki} - tx & \text{if } m_k \text{ is chosen} \\ 0 & \text{if none is chosen,} \end{cases} \tag{21}$$

where x is the location on the unit interval $[0, 1]$.

Each consumer with type $(\tilde{c}, x)$ chooses the utility-maximizing services among the menus of two firms, or chooses to buy nothing.

As in the previous section, suppose $a_{ki} \leq a_{k+1i}$ and $p_{ki} \geq p_{k+1i}$ for each $k = 0, \ldots, K-1$ and $i = 1, 2$.

In general, there might be numerous patterns of menus for a firm which will be a best response to a menu of the other firm. Although it is interesting, it is not easy to figure out these general patterns of Nash equilibrium pair of menus. Thus, I restrict an attention to the symmetric equilibrium, where $M_1 = M_2 = M = (m_k)_{k=0}^K$. In this simplified case, we can see that $c_{k1} = c_{k2} = c_k$ for each k, where c_{ki} is the type a consumer who is indifferent between m_{ki} and m_{k-1i}.

Suppose $(m_k)_{k=0}^K$ is the symmetric equilibrium menu. For simplicity, I assume that v is sufficiently large relative to C and t, so that in any equilibrium consumer chooses some goods. I also assume that the second-order condition for the equilibrium menu pricing is satisfied[12] Under the setting described above, I obtain the following result which shows that freemium remains to be an equilibrium even if we take competition into account.

Proposition 3.1. *The symmetric equilibrium menu is freemium.*

Proof. In Apenndix. □

Next, as in Section 2, I consider the properties of equilibrium freemium strategies. Deriving the first-order conditions at the symmetric equilibrium I obtain the following result.

Proposition 3.2. *Symmetric equilibrium menu* (a_B, p_B), (a_P, p_P)*, and the price for advertisers are determined by the following equations:*

$$p_a - c = \frac{1 - F(c)}{f(c)} \left(\frac{p_P + a_P p_a}{t} - 1 \right) \tag{22}$$

$$= \frac{F(c)}{f(c)} \left(1 - \frac{p_B + a_B p_a}{t} \right) \tag{23}$$

$$= \frac{1}{\eta_P(c)} \left(\frac{\hat{c}_P}{t} (p_P + a_P p_a) - p_a \right) \tag{24}$$

$$= \frac{1}{\eta_B(c)} \left(p_a - \frac{1 - G(p_a)}{g(p_a)} - \frac{\hat{c}_B}{t} (p_B + a_B p_a) \right) \tag{25}$$

where $\eta_B(c) \equiv \frac{f(c)c}{F(c)}$ *and* $\eta_P(c) \equiv \frac{f(c)c}{1-F(c)}$ *are the surplus elasticity of demand for each goods, and* $\hat{c}_B \equiv E[\tilde{c}|\tilde{c} \leq c]$ *and* $\hat{c}_P \equiv E[\tilde{c}|\tilde{c} \geq c]$ *are the average type of consumers who choose basic good and premium good, respectively.*

Proof. In Appendix. □

From the equation (22) and the equation (23), we can see that if $p_a \geq c$, then the average revenue from premium users $p_P + a_P p_a$ is higher than the average revenue from basic users $p_B + a_B p_a$ and vice versa. Thus, the relation between the values of threshold types determines the contributions of premium and basic consumers to the platforms' revenues.

As a corollary of the proposition above, I find that the equilibrium profit is independent of any fundamentals of consumers or advertisers.

Corollary 3.1. *In the symmetric equilibrium, the resulting profit for each firm is* $\frac{t}{2}$.

This result is somewhat striking because all the benefits for platforms from adopting freemium disappears once a competition is introduced. This property is known as *revenue neutrality property* that if all consumers are served, then any exogenous increase in revenues per consumers are competed away (Anderson and Gabszewicz 2006, Armstrong 2006).

The next corollary shows the determinant of the transaction fees on advertisers.

[12] I do not derive the second-order condition but just assume that the second-order condition is satisfied since it is not the main focus of this paper. Zennyo (2016) derives the second-order condition under a similar specification to my model.

Corollary 3.2. *In the symmetric equilibrium, the price charged on advertisers is determined by following formula:*

$$p_a = F(c)\left(\frac{\hat{c}_B}{t}(p_B + a_B p_a) + \frac{1 - G(p_a)}{g(p_a)}\right) + (1 - F(c))\frac{\hat{c}_P}{t}(p_P + a_P p_a). \tag{26}$$

Equation (26) shows that platforms choose its transaction fees p_a to be the weighed average of marginal loss terms from basic consumers and premium consumers.

As seen above, the main property that the platform segment consumers into those who view fewer advertisements and those who view all advertisements holds even under duopoly. This result indicates the robustness of freemium pricing in different situations.

4. Conclusion

Freemium with advertisement is so prevalent that any people who have ever used online applications have faced a choice between free service with a lot of advertisements and ad-free premium service. I examine the optimality of this business model, and show that under certain specifications which are naturally adopted in the literature of two-sided markets, freemium with advertisements is actually the best way to collect revenues from both consumer and advertisers. The property that at the optimal nonlinear pricing, there are two bunches of consumers at the top and at the bottom seems to be robust to several modifications of specifications.

One possible direction of future research is the analysis of a platform who uses quality and advertisements as instruments of price discrimination at the same time. In reality, platforms use not only the amount of advertisements but also the qualities of services as instruments of price discrimination. There might be an interesting interaction when we analyze these things together. In addition, there might be heterogeneity in the externality of each agent in one side on agents on another side, as in Gomes and Pavan (2016) and Jeon et al. (2016). Incorporating these elements may make other differences, such as complementarities or substitutabilities in quality and the amount of advertisements. Also, analyzing asymmetric equilibria under duopoly menu competition might be interesting since in the real world, different business models coexist and this cannot be explained by my simple model. However, just computing asymmetric equilibria using my model ends up being a tedious calculation without meaningful results. Thus, one have to invent more tractable framework to tackle with this problem.

Appendix

A. Proofs

A.1. Proof of Proposition 2.1

I introduce several lemmata and use them to prove the proposition.

Lemma A.1. *Optimal menu pricing satisfies* $a_0 = 0$, $p_0 = v$.

Proof. Let $(m_k)_{k=0}^K$ be an optimal menu and suppose that $a_0 > 0$. By the first-order condition for p_k, $k = 1, \dots, K-1$, we have $c_k = c_{k+1}$ for $k = 1, \dots, K-1$. In this case, the profit can be written as

$$d_0 p_0 + d_K p_K + (d_0 a_0 + d_K a_K) G^{-1}(1-a)).$$

Consider a menu $(m'_k)_{k=0}^K$ such that

$$\begin{aligned} a'_0 &= 0, \quad p'_0 = v \\ a'_1 &= a_0, \quad p'_1 = p_0 \\ a'_k &= a_k, \quad p'_k = p_k \text{ for } k = 2, \dots, K. \end{aligned}$$

We can see that this menu obtains the profit

$$d_0 p_0 + d_K p_K + (d_0 a_0 + d_K a_K) G^{-1}(1-a) + (1 - F(c_0))v,$$

which is higher than the profit obtained by $(m_k)_{k=0}^K$, violating the optimality. Thus, we must have $a_0 = 0$.

Next, suppose that $(m_k)_{k=0}^K$ satisfies $a_0 = 0$ but $p_0 < v$. In this case, $c_1 = \frac{p_0 - p_1}{a_1}$. Then increasing p_k by small ε for all k does not change c_k for all k. Thus, this price change increases the profit by $\sum_{k=0}^K d_k \varepsilon > 0$. This contradicts the optimality of $(m_k)_{k=1}^K$. Thus, we must have $p_0 = v$. □

Lemma A.2. *Optimal menu pricing satisfies* $c_1 = \cdots = c_K$.

Proof. By the proof of Lemma A.1, we can see that $c_1 = \frac{v - p_1}{a_1}$ and $c_2 = \cdots = c_K$. Thus, what remains to be shown is that $c_1 = c_2$. To see this, consider the first-order condition for p_1:

$$\frac{\partial d_0}{\partial p_1}(p_0 + a_0 G^{-1}(1-a)) + \frac{\partial d_1}{\partial p_1}(p_1 + a_1 G^{-1}(1-a)) + \frac{\partial d_2}{\partial p_1}(p_2 + a_2 G^{-1}(1-a)) + d_1 = 0.$$

This equation can be rewritten as

$$f(c_1)(c_1 - G^{-1}(1-a)) + F(c_1) = f(c_2)(c_2 - G^{-1}(1-a)) + F(c_2),$$

and $c_1 = c_2$ satisfies this condition. □

Lemma A.3. *Optimal menu pricing satisfies* $a_K = a$

Proof. By Lemma A.1 and Lemma A.2, the profit maximization problem is reduced to

$$\begin{aligned} \max_{a_K, a, p} \quad & \Pi = (1 - F(c))v + F(c)(p + a_K G^{-1}(1-a)) \\ \text{s.t.} \quad & c = \frac{v - p}{a_K} \\ & a_K \leq a. \end{aligned}$$

In this case, as long as $a_K < a$, the platform can increase the profit by reducing a. Thus, $a_K = a$. □

Summarizing these lemmata, we obtain the first statement.

Next, the profit maximization problem (6) can be rewritten as

$$\begin{aligned} \max_{a,p} \quad & (1 - F(c))v + F(c)(p + aG^{-1}(1-a)) \\ \text{s.t.} \;\; & c = \frac{v-p}{a}. \end{aligned} \tag{27}$$

Proof. The first-order condition for p is given by

$$f(c)(c - G^{-1}(1-a)) + F(c) = 0.$$

Rearranging this equation, we obtain

$$p_a - c = \frac{F(c)}{f(c)}.$$

The first-order condition for a is given by

$$cf(c)(c - G^{-1}(1-a)) + F(c)(G^{-1}(1-a) - aG^{-1'}(1-a)) = 0.$$

Applying the inverse function theorem, substituting the first-order condition for p, and rearranging, we obtain

$$p_a - c = \frac{1 - G(p_a)}{g(p_a)}.$$

Finally, we can see that the second-order condition is satisfied when $\frac{F}{f}$ is increasing and $\frac{1-G}{g}$ is decreasing.

Putting these together, we obtain the Proposition 2.1. □

A.2. Proof of Result 2.1

Proof. First, consider the case where F is replaced by $\tilde{F}$ which dominates F according to reverse hazard rate. Then, we have $\frac{\tilde{F}(\tilde{c})}{\tilde{f}(\tilde{c})} \leq \frac{F(\tilde{c})}{f(\tilde{c})}$ for any $\tilde{c} \in [0, C]$. Let c' and p'_a be the threshold types under $\tilde{F}$ and c and p_a be the threshold types under F. I first show that $p'_a \geq p_a$. Suppose that $p_a > p'_a$. Then

$$c = p_a - \frac{1 - G(p_a)}{g(p_a)} > p'_a - \frac{1 - G(p'_a)}{g(p'_a)} = c'.$$

Combining equations (7) and (8), we obtain

$$\frac{1 - G(p_a)}{g(p_a)} = \frac{F(c)}{f(c)}, \text{ and } \frac{1 - G(p'_a)}{g(p'_a)} = \frac{\tilde{F}(c')}{\tilde{f}(c')}.$$

Putting these together, we obtain

$$\frac{1 - G(p_a)}{g(p_a)} = \frac{F(c)}{f(c)} \geq \frac{F(c')}{f(c')} \geq \frac{\tilde{F}(c')}{\tilde{f}(c')} = \frac{1 - G(p'_a)}{g(p'_a)},$$

which leads to $p_a \leq p'_a$ since $\frac{1-G}{g}$ is decreasing and derives contradiction. Thus, we have $p'_a \geq p_a$ and $c' \geq c$ follows.

Next, I show that p_a and c are increasing in θ in the second case. If the inverse hazard rate is characterized by $\lambda(\theta, b)$, then the first-order conditions can be written as

$$c + \frac{F(c)}{f(c)} - p_a = 0$$
$$p_a - \lambda(\theta, p_a) - c = 0.$$

Differentiating these equations by θ and rearranging, we can see that

$$\frac{dp_a}{d\theta} = \frac{-\frac{\partial \lambda(\theta, p_a)}{\partial \theta} \frac{\partial}{\partial c}\left(c + \frac{F(c)}{f(c)}\right)}{\left(1 - \frac{\partial}{\partial p_a}\left(p_a - \lambda(\theta, p_a)\right) \frac{\partial}{\partial c}\left(c + \frac{F(c)}{f(c)}\right)\right)} \geq 0$$

Thus, p_a is increasing in θ. Then the fact that c is increasing in θ is straightforward.

□

A.3. Proof of Proposition 2.3

Proof. Let c' and p'_a be the threshold types under ad-supported model and c and p_a be the threshold under freemium. Suppose that $c' < c$. Then $p'_a < p_a$ is also follows from the equation (8) and the equation (19). Combining the first-order condition under each business model and assumption above, we obtain

$$\frac{1 - G(p'_a)}{g(p'_a)} < \frac{F(c')}{f(c')} \leq \frac{F(c)}{f(c)} = \frac{1 - G(p_a)}{g(p_a)}.$$

Thie inequality requires $p'_a > p_a$ to hold, which contradicts $p'_a < p_a$. Thus, we must have $c' \geq c$ and $p'_a \geq p_a$. □

A.4. Proof of Proposition 3.1

First, I show a lemma similar to Lemma A.1.

Proof.

Lemma A.4. *In the symmetric equilibrium, $c_k = c_{k+1}$ for $k = 1, \dots K - 1$.*

Proof. Consider the incentive to deviate from the equilibrium strategies. Suppose that firm 1 changes p_k by $\varepsilon > 0$. Then, c_k and c_{k+1} change to c_k^ε and c_{k+1}^ε where

$$c_k^\varepsilon = \frac{p_{k-1} - p_k - \varepsilon}{a_k - a_{k-1}}, \quad c_{k+1}^\varepsilon = \frac{p_k - p_{k+1} + \varepsilon}{a_{k+1} - a_k}.$$

Thus the consumer with type $\tilde{c} \in [c_k^\varepsilon, c_k]$ will turn to prefer (a_{k-1}, p_{k-1}) to $(a_k, p_k + \varepsilon)$, and the consumer with type $\tilde{c} \in [c_{k+1}, c_k^\varepsilon]$ will turn to prefer (a_{k+1}, p_{k+1}) to $(a_k, p_k + \varepsilon)$.

$$\hat{x}_{k-1}(\tilde{c}) = \frac{1}{2} - \frac{(p_{k-1} - p_k) - \tilde{c}(a_k - a_{k-1})}{2t}$$
$$\hat{x}_k = \frac{1}{2} - \frac{\varepsilon}{2t}$$
$$\hat{x}_{k+1}(\tilde{c}) = \frac{1}{2} - \frac{(\tilde{c}(a_{k+1} - a_k) - (p_k - p_{k+1}))}{2t}.$$

Finally, the profit which is affected from ε is

$$\int_{c_k^\varepsilon}^{c_k} \hat{x}_{k-1}(\tilde{c})f(\tilde{c})d\tilde{c}\left(p_{k-1} + a_{k-1}G^{-1}(1-a)\right) + \int_{c_{k+1}^\varepsilon}^{c_k^\varepsilon} \hat{x}_k f(\tilde{c})d\tilde{c}\left(p_k + \varepsilon + a_k G^{-1}(1-a)\right)$$
$$+ \int_{c_{k+1}}^{c_{k+1}^\varepsilon} \hat{x}_{k+1}(\tilde{c})f(\tilde{c})d\tilde{c}\left(p_{k+1} + a_{k+1}G^{-1}(1-a)\right). \tag{28}$$

Taking derivative with respect to ε at $\varepsilon = 0$, we obtain the first-order condition

$$\begin{aligned} &f(c_k)(c_k - G^{-1}(1-a)) - f(c_{k+1})(c_{k+1} - G^{-1}(1-a)) \\ &\quad - \frac{1}{t}(F(c_k) - F(c_{k+1}))(p_k + a_k G^{-1}(1-a)) + (F(c_k) - F(c_{k+1}) = 0. \end{aligned}$$

Then we can see that $c_k = c_{k+1}$ satisfies the first-order condition. Thus, as in the previous section, we have $c_k = c_{k+1}$ for $k = 1, \ldots, K-1$.

□

What remains to be shown is that $c_1 = c_2$, which can be shown in an analogous procedure to the above. □

A.5. Proof of Proposition 3.2

Proof. To show the first equation, consider the symmetric equilibrium $(a_B, p_B)(a_P, p_P))$ with threshold type c. Suppose that the platform 1 deviates its equilibrium premium price to $p_P + \varepsilon$. Then, the threshold locations for each type c is given by

$$\hat{x}(\tilde{c}) = \begin{cases} \frac{1}{2} - \frac{\varepsilon}{2t} & \text{if } \tilde{c} \geq c^\varepsilon \\ \frac{1}{2} - \frac{(p_P - p_B) - (a_B - a_P)c}{2t} & \text{if } \tilde{c} \in [c, c^\varepsilon) \\ \frac{1}{2} & \text{if } \tilde{c} < c, \end{cases} \tag{29}$$

where $c^\varepsilon = \frac{p_P - p_B - \varepsilon}{a_B - a_P}$. Then, the profit for the platform is given by

$$\begin{aligned} &\int_{c^\varepsilon}^{C} \hat{x}(\tilde{c})f(\tilde{c})d\tilde{c}(p_P + \varepsilon + a_P p_a) + \int_{c}^{c^\varepsilon} \hat{x}(\tilde{c})f(\tilde{c})d\tilde{c}(p_B + a_B p_a) \\ &\quad + \frac{1}{2}F(c)(p_B + a_B p_a). \end{aligned} \tag{30}$$

Then, differentiating by ε at $\varepsilon = 0$, we obtain the first-order condition

$$\frac{1}{2}(1 - F(c)) - \frac{1}{2t}(1 - F(c))(p_P + a_P p_a) - f(c)(c - p_a) = 0. \tag{31}$$

Rearranging this, we obtain the first equation. The second equation is obtained in an analogous way.

Next, consider an incentive to increase a_P. Consider the case where the platform 1 increases a_P by ε. Then, the threshold locations for each type c is given by

$$\hat{x}(\tilde{c}) = \begin{cases} \frac{1}{2} - \frac{\varepsilon\tilde{c}}{2t} & \text{if } \tilde{c} \geq c^{\varepsilon} \\ \frac{1}{2} - \frac{(p_P - p_B) - (a_B - a_P)\tilde{c}}{2t} & \text{if } \tilde{c} \in [c, c^{\varepsilon}) \\ \frac{1}{2} & \text{if } \tilde{c} < c, \end{cases} \tag{32}$$

where $c^{\varepsilon} = \frac{p_P - p_B}{a_B - a_P - \varepsilon}$. Then, the profit for the platform is given by

$$\begin{aligned} &\int_{c^{\varepsilon}}^{C} \hat{x}(\tilde{c}) f(\tilde{c}) d\tilde{c} (p_P + \varepsilon + a_P p_a) + \int_{c}^{c^{\varepsilon}} \hat{x}(\tilde{c}) f(\tilde{c}) d\tilde{c} (p_B + a_B p_a) \\ &+ \frac{1}{2} F(c)(p_B + a_B p_a). \end{aligned} \tag{33}$$

Differentiating by ε at $\varepsilon = 0$, we obtain the first-order condition

$$\begin{aligned} \frac{(1 - F(c))}{2} p_a - \frac{\int_c^C \tilde{c} f(\tilde{c}) d\tilde{c}}{2t} (p_P + a_P p_a) - \frac{f(c)}{2} c(c - p_a) = \\ \frac{(1 - F(c))}{2} p_a - \frac{(1 - F(c)) E[\tilde{c} | \tilde{c} \geq c]}{2t} (p_P + a_P p_a) - \frac{f(c)}{2} c(c - p_a) = 0. \end{aligned} \tag{34}$$

Rearranging this, we obtain the third equation. The fourth equation can be derived in a similar way.

□

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