Randomized Strategy Equilibrium and Bertrand Equilibrium in the Action Commitment Game with a Small Cost of Leading

Toshihiro Matsumura  
Institute of Social Science, University of Tokyo  
Takeshi Murooka  
Department of Economics, University of California, Berkeley  
and  
Akira Ogawa*  
College of Liberal Arts, International Christian University  
March 5, 2010

Abstract

We introduce a small cost of leading in the two-player action commitment game formulated by Hamilton and Slutsky (1990). We investigate a price competition model and find that any randomized strategy equilibria converge to the Bertrand equilibrium.

JEL classification numbers: L13, C72  
Key words: endogenous timing, gain for waiting, mixed strategy equilibria

*Corresponding author: Akira Ogawa, College of Liberal Arts, International Christian University, 3-10-2, Osawa, Mitaka-shi, Tokyo 181-8585, Japan. Phone: (81)-422-33-3169. Fax: (81)-422-34-6982. E-mail: ogawaa@icu.ac.jp
1 Introduction

In their pioneering work, Hamilton and Slutsky (1990) formulated two models of the endogenous timing game: observable delay game and action commitment game. Both models have been intensively used as frameworks for analyzing the endogenous role in many subsequent works. In the action commitment game, they showed that three pure strategy equilibria exist: two sequential-move outcomes (either player 1 or 2 becomes the Stackelberg leader) and one simultaneous-move outcome (both players act in the first period such as in the Cournot model and the Bertrand model). They also emphasized that the simultaneous-move outcome is less plausible because the two sequential-move equilibria are the only pure strategy equilibria in undominated strategies and the simultaneous-move outcome is supported by weakly dominated strategies.

Further, many other works have shown that the simultaneous-move outcome is vulnerable. Mailath (1993) and Normann (1997) investigated incomplete information games wherein waiting players obtain additional information on demand or cost. They showed that while a slight informational gain from waiting eliminates the simultaneous-move equilibrium where both players act in the first period, it does not eliminate the two sequential-move outcomes. Robson (1990) and Matsumura (1999) showed the same result in complete information games with small inventory costs. The results of the above studies showed that the set of equilibrium outcomes is not lower hemi-continuous with respect to the cost of leading and that the simultaneous-move outcome is an equilibrium only when the cost is strictly zero. This may indicate that the simultaneous-move outcome (such as Bertrand and Cournot) is less plausible in the endogenous timing game than the sequential-move outcomes (such as Stackelberg). However, recent studies on laboratory experiments have suggested the opposite. Huck et al. (2002) and Fonseca et al. (2005) reported that the Stackelberg outcomes are rarely observed, while the simultaneous-move outcome occurs frequently.

We bridge the gap between the contrasting theoretical and empirical results by investigating a randomized strategy equilibrium. We introduce a small cost of leading into the action commitment game with price competition. In the action commitment game with small leading costs, there are two pure strategy equilibria (either player 1 or 2 is the leader). Naturally, we expect that a randomized strategy equilibrium also exists. In fact, we find that a randomized strategy equilibrium does exist and that it converges to the simultaneous-move outcome when the cost of leading converges to zero. Our result indicates that the simultaneous-move outcome is not as vulnerable as the current theoretical literature asserts. Indeed, we can regard the simultaneous-move equilibrium as a degenerate randomized strategy equilibrium.

Our study is closely related to Pal (1996). He investigated a two-production-period model formulated by
Saloner (1987) and showed a similar result. Although he did not discuss the action commitment game, we can apply his principle to this game under the assumptions he made. However, there is one important difference: his result depends on the assumption of strategic substitutes. If the reaction curve in the simultaneous-move game is upward-sloping, no randomized strategy equilibrium exists in the two-production-period model. Thus, his analysis is not applicable to the strategic complement cases. We investigate a price competition model with strategic complements and show that in the action commitment game, randomized strategy equilibria exist, and any of them converges to the Bertrand equilibrium as the cost of leading goes to zero.

2 The model

We consider the action commitment game of Hamilton and Slutsky (1990), where firms first choose the timing of their actions. The game is a complete information game. There are two possible time stages for price choice: in the first stage, firm $i$ independently chooses its price $p_i$, or waits until the next stage. At the beginning of the second stage, each firm observes the rival’s price chosen in the first stage. In the second stage, only those firms that have been waiting until the second stage choose their prices.

The payoff of firm $i$ ($i, j = 1, 2, i \neq j$) is given by $V_i(p_i, p_j) - \varepsilon_i$ if firm $i$ chooses the first period, and by $V_i(p_i, p_j)$ if it chooses the second period, where $\varepsilon_i$ is a positive constant and it indicates the cost of leading of firm $i$. We assume that $V_i(p_i, p_j)$ is strictly concave, continuously differentiable, and increasing in $p_j$. We also assume that $|\partial^2 V_i(p_i, p_j)/\partial^2 p_i| > |\partial^2 V_i(p_i, p_j)/\partial p_i \partial p_j|$.\footnote{This is the so-called stability condition. These assumptions are standard in the field. See Vives (1985).}

3 Fixed Timing Games

Before discussing the endogenous timing game, we investigate fixed timing games (simultaneous-move and sequential-move games). We assume an interior solution in these fixed timing games.

3.1 Bertrand game

Consider the simultaneous-move game where firm $i$ ($i = 1, 2$) independently chooses $p_i$. The first order condition is given by $\partial V_i(p_i, p_j)/\partial p_i = 0$ ($i = 1, 2, i \neq j$). Let $(p_1^B, p_2^B)$ be the Bertrand equilibrium prices.

\footnote{Pal (1991) introduced cost differences between two periods into the model of Saloner (1987) and showed that the existence of a leading cost changes the set of equilibria.}
3.2 Stackelberg game

Consider the sequential-move game where firm $i$ chooses $p_i$ first, and then firm $j$ chooses $p_j$ after observing $p_i$ $(i, j = 1, 2, i \neq j)$. Let $R_j(p_i)$ be the reaction function of firm $j$. We assume that $R_j(p_i)$ is continuously differentiable and $R'_j > 0$. Firm $i$ maximizes $V_i(p_i, R_j(p_i)) - \varepsilon_i$ with respect to $p_i$. The first-order condition is

$$\frac{\partial V_i(p_i, p_j)}{\partial p_i} + \frac{\partial V_i(p_i, p_j)}{\partial p_j} dR_j(p_i) dp_i = 0.$$ 

We assume that $V_i(p_i, R_j(p_i))$ is strictly concave. Let superscript “L” and “F” denote the equilibrium outcome of the leader and the follower in the Stackelberg game respectively. Let $(p^L_i, p^F_j)$ be the equilibrium prices in the sequential-move game. We assume that $p^L_i > p^F_i$. This always holds under the above assumptions if the two firms are symmetric. Needless to say, the symmetry of the firms is not a necessary condition.

4 Results

We now investigate the action commitment game. First, we present two well-known results on pure strategy equilibria in the general context of the action commitment game. Denote $\bar{\varepsilon}_i = V_i(p^L_i, p^F_j) - V_i(p^B_i, p^B_j)$. It can be shown that $\bar{\varepsilon}_i$ is strictly positive in our setting.

Result 1 (Hamilton and Slutsky 1990) Suppose that $\varepsilon$ is zero. Three pure strategy equilibria exist: one is the Bertrand equilibrium where both firms choose the first period; the other two are Stackelberg equilibria where one firm chooses the first period and the other chooses the second period.

Result 2 Suppose that $0 < \varepsilon_i < \bar{\varepsilon}_i$ $(i = 1, 2)$. There are only two pure strategy equilibria and both are Stackelberg equilibria.

We now present our result. We discuss the randomized strategy equilibrium.

**Proposition** Suppose that $0 < \varepsilon_i < \bar{\varepsilon}_i$ $(i = 1, 2)$. (i) For any $\varepsilon_1, \varepsilon_2$, a randomized strategy equilibrium exists. (ii) Take any sequence of randomized strategy equilibria where firm $i$’s cost of leading is $\varepsilon^n_i$ and firm $i$ chooses the first stage with probability $q^n_i$. If $\lim_{n \to \infty} \varepsilon^n_i = 0$, then $\lim_{n \to \infty} q^n_i = 1$. If $\lim_{n \to \infty} \varepsilon^n_i = \bar{\varepsilon}_i$, then $\lim_{n \to \infty} q^n_i = 0$.

**Proof.** See Appendix.

Although the set of pure strategy equilibria is discontinuous at $\varepsilon = 0$, the set of equilibria including mixed strategy is not. We can regard the Bertrand equilibrium when $\varepsilon = 0$ as a degenerate randomized strategy equilibrium. Our result indicates that the simultaneous-move game is not so vulnerable if we consider the randomized strategy equilibrium. We believe that our result bridges the gap between theoretical and empirical results mentioned in Introduction.
Appendix

Proof of Proposition (i) Suppose that firm $i$ chooses to name its price in the second stage. At the beginning of the second stage, firm $i$ observes whether or not firm $j$ named its price in the first stage. When firm $j$ names its price in the first stage, firm $i$ chooses $p_i = R_i(p_j)$. When firm $j$ chooses to name its price in the second stage, the firms face Bertrand competition. As a result, firms choose $(p_i, p_j) = (p_i^B, p_j^B)$.

Suppose that firm $i$ chooses to name its price in the first stage. Let $p_i^1$ be the price of firm $i$ when it chooses to name its price in the first stage. Further, let $p_j^2$ be the price of firm $j$ when firm $i$ chooses $p_i^1$ in the first stage. Let $q_j$ be the probability that firm $j$ chooses to name its price in the first stage. Firm $i$ chooses $p_i^1$ so as to maximize

$$q_j V_i(p_i^1, p_j^1) + (1 - q_j) V_i(p_i^1, R_j(p_i^1)) - \varepsilon_i.$$

The first-order condition is

$$q_j \frac{\partial V_i(p_i^1, p_j^1)}{\partial p_i^1} + (1 - q_j) \left[ \frac{\partial V_i(p_i^1, p_j^2)}{\partial p_i^1} + \frac{\partial V_i(p_i^1, p_j^2)}{\partial p_j^2} \cdot \frac{dR_j(p_i^1)}{dp_i^1} \right] = 0. \quad (1)$$

In the randomized strategy equilibrium, the expected payoff of firm $i$ when it chooses to name its price in the first stage must be equal to its expected payoff when it chooses to name its price in the second stage. Thus,

$$q_j V_i(p_i^1, p_j^1) + (1 - q_j) V_i(p_i^1, R_j(p_i^1)) - \varepsilon_i = q_j V_i(R_i(p_i^1), p_j^1) + (1 - q_j) V_i(p_i^B, p_j^B). \quad (2)$$

Substituting $i = 1, j = 2$ and $i = 2, j = 1$ into (1) and (2), we have four equations. Since we assume $|\partial^2 V_i(p_i, p_j) / \partial p_i | > |\partial^2 V_i(p_i, p_j) / \partial p_i \partial p_j|$, we can apply the implicit function theorem to the two equations in (1). Thus we have the solution $p_i^1(q_i, q_j), p_j^1(q_i, q_j)$, which is continuously differentiable for all $q_i, q_j \in [0, 1]$.

Substituting $p_i^1(q_i, q_j), p_j^1(q_i, q_j)$ into (2), we have

$$q_j [V_i(p_i^1(q_i, q_j), p_j^1(q_i, q_j)) - V_i(R_i(p_j^1(q_i, q_j)), p_j^1(q_i, q_j))] + (1 - q_j) [V_i(p_i^1(q_i, q_j), R_j(p_j^1(q_i, q_j))) - V_i(p_i^B, p_j^B)] = \varepsilon_i. \quad (3)$$

Note that if $q_j = 0$, then $p_i^1(q_i, q_j) = p_i^F$ and the left-hand side of (3) equals $\bar{\varepsilon}_i$. Note also that if $q_j = 1$, then $p_j^1(q_i, q_j) = R_i(p_j^1)$ and the left-hand side of (3) equals zero. Hence, the left-hand side of (3) is greater (smaller) than the right-hand side of (3) when $q_j = 0 (q_j = 1)$. Thus, we can apply the fixed point theorem shown in Garcia and Zangwill (1979). Under these conditions, for any $\varepsilon_i, \varepsilon_j$ there exists $(q_i^*, q_j^*)$ that comprises a randomized strategy equilibrium.

Proof of Proposition (ii) First, we prove that $\lim_{n \to \infty} q_j^n = 1$ if $\lim_{n \to \infty} \varepsilon_i^n = 0$. We prove it by contradiction. Suppose not. Then, there exists a sequence of randomized strategy equilibrium such that $\{\varepsilon_i^n\}_{n=1}^\infty$ converges to 0 and $\{q_j^n\}_{n=1}^\infty$ does not converge to 1.

---

3 This is an n-dimensional version of the intermediate value theorem.
Let $q_j^0$ be $\lim inf_{n \to \infty} q_j^n$. Since $q_j^n$ does not converge to 1, we have $q_j^0 < 1$. When $q_j = 0$, $p_i^1(q_i, q_j) = p_i^L$, and the left-hand side of (3) is equal to $\bar{V}_i$. Then, (3) is not satisfied, a contradiction. Thus, $q_j^0 > 0$. Let $\{q_j^n\}_{n=1}^{\infty}$ be a subsequence of $\{q_j^n\}_{n=1}^{\infty}$ which converges to $q_j^0$. Note that $\{\varepsilon_i^n\}_{n=1}^{\infty}$ converges to 0. By the continuity of $p_i^1, p_i^2, R_j,$ and $V_i$,

$$\lim_{k \to \infty} \left[ q_j^k \left\{ V_i(p_i^1(q_i^k, q_j^k), p_i^1(q_i^k, q_j^k)) - V_i(R_i(p_j^1(q_i^k, q_j^k)), p_j^1(q_i^k, q_j^k)) \right\} \right.$$

$$+ (1 - q_j^k) \left\{ V_i(p_i^1(q_i^k, q_j^k), R_j(p_i^1(q_i^k, q_j^k))) - V_i(p_i^B, p_j^B) \right\} \right]$$

$$= q_j^0 \left\{ V_i(p_i^1(q_i^0, q_j^0), p_i^1(q_i^0, q_j^0)) - V_i(R_i(p_j^1(q_i^0, q_j^0)), p_j^1(q_i^0, q_j^0)) \right\} + (1 - q_j^0) \left\{ V_i(p_i^1(q_i^0, q_j^0), R_j(p_i^1(q_i^0, q_j^0))) - V_i(p_i^B, p_j^B) \right\}$$

$$= \lim_{k \to \infty} \varepsilon_i^k = 0$$

holds from (3). Thus, $q_j^0 \in (0, 1)$ and $\varepsilon_i^0 = 0$ comprise the randomized strategy equilibrium. Hence in the equilibrium, (5) must be equal to (6), where (5) and (6) are expected payoffs when firm $i$ chooses the first and the second periods respectively.

$$q_j^0 V_i \left( p_i^1(q_i^0, q_j^0), p_j^1(q_i^0, q_j^0) \right) + (1 - q_j^0) V_i \left( p_i^1(q_i^0, q_j^0), R_j(p_j^1(q_i^0, q_j^0)) \right)$$

$$q_j^0 V_i \left( R_i(p_j^1(q_i^0, q_j^0)), p_j^1(q_i^0, q_j^0) \right) + (1 - q_j^0) V_i \left( R_i(p_j^1(q_i^0, q_j^0)), p_j^1(q_i^0, q_j^0) \right)$$

We show that (5) > (6), and derive a contradiction. Suppose that firm $i$ chooses $p_i^1 = R_i(p_j^1(q_i^0, q_j^0))$ in the first period rather than $p_i = p_i^1(q_i^0, q_j^0)$. This price may not optimal for firm $i$. Thus, (5) ≥ (7).

$$q_j^0 V_i \left( R_i(p_j^1(q_i^0, q_j^0)), p_j^1(q_i^0, q_j^0) \right) + (1 - q_j^0) V_i \left( R_i(p_j^1(q_i^0, q_j^0)), R_j(p_j^1(q_i^0, q_j^0)) \right)$$

We show that (7) > (6). Since the first term in (7) is equal to the first term in (6), it is sufficient to show that the second term in (7) is strictly larger than the second term in (6). For any $q_j \in (0, 1)$, $p_i^1(q_i, q_j)$ must be in $(p_i^B, p_i^L)$. Since $p_i^B < p_i^1(q_i^0, q_j^0)$ and $R_j^0 > 0$, we have $p_i^B < R_j(p_i^1(q_i^0, q_j^0))$. Because $p_j^F = R_j(p_i^L)$ and $p_i^1(q_i^0, q_j^0) < p_i^L$, we get $R_j(p_i^1(q_i^0, q_j^0)) < p_j^F$. By the assumption of $p_i^1 < p_i^L$, we have $R_j(p_i^1(q_i^0, q_j^0)) < R_j(p_i^1(q_i^0, q_j^0))$. Since $V_i(p_i, R_j(p_i))$ is strictly concave, we have that $V_i(p_i, R_j(p_i))$ is increasing in $p_i$ for $p_i \in (p_i^B, p_i^L)$. Because we showed that $R_j(p_i^1(q_i^0, q_j^0)) \in (p_i^B, p_i^L)$, we have $V_i(R_i(p_j^1(q_i^0, q_j^0)), R_j(p_j^1(q_i^0, q_j^0))) > V_i(p_i^B, R_j(p_i^1)) = V_i(p_i^B, p_j^B)$. Therefore, (5) ≥ (7) > (6) holds, a contradiction.

Finally, we prove the last part of Proposition. $\lim_{n \to \infty} q_j^n = 0$ if $\lim_{n \to \infty} \varepsilon_i^n = \bar{\varepsilon}_i$. The first term in (3) is non-positive and the second term in (3) is at largest $(1 - q_j)\bar{\varepsilon}_i$. Thus, $q_j \to 0$ as $\varepsilon_i \to \bar{\varepsilon}_i$. \hfill \Box

---

4 This was suggested in Hamilton and Slutsky (1990) and was shown in Pastine and Pastine (2004).
References


