Vote Weight Disparity with Endogenous Information: A Note*

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Abstract

A large society is confronted with a dichotomous choice. There are predetermined weights with which the members' votes are summed up and the society chooses the alternative that wins the plurality. The members' information about the alternatives are endogenous: they invest some efforts before making their votes, and the levels of the investments determine the accuracies of their votes. We find that removing off the existing vote weight disparity could deteriorate the society's performance, but whether it would occur or not can be figured out by just checking the sign of some covariances.

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1 Introduction

In collective decision processes, there are sometimes essential vote weight disparities among members (maybe due to historical reasons). Once the disparity is recognized as unignorably large, the correction becomes public concern. The correction would remedy inequity among members. However, once considering (i) the role of the collective decision process for aggregating dispersed information among members and (ii) the existence of their efforts to gather such information (or cast their votes more carefully), another aspect arises: it would also change their incentives to make such efforts, which might result in the worse performance of the collective decision.

To approach this aspect, we develop an election model with vote weight disparities and accuracy improvement costs, and investigate its asymptotic properties. (Theorem.) We find that (i) the performance could be damaged by removing off the existing disparity, but (ii) whether it would happen or not can be simply figured out by just checking the sign of some covariances. (Corollaries 1 and 2.)

The next section presents the model, and the results and discussions are in Section 3. The proof of Theorem is in Appendix.

2 Model

There is a set of voters, $N_n = \{1, 2, ..., |N_n|\}$, the size of which is an odd number and $\lim_{n\to\infty} |N_n| = \infty$. The strategy of each voter $i \in N_n$ is the accuracy of his vote, $q_i \in [1/2, 1]$, for which he incurs the investment costs of $C_i(q_i)$. $C_i(q)$ is strictly increasing, strictly convex and twice continuously differentiable in q, and $C(1/2) = 0$. If the society succeeds in adopting the right policy, he receives the utility r_i. Otherwise, he receives 0.

The policy is chosen between two (symmetric) alternatives by majority vote, where the vote of each voter i is multiplied by an integer m_i: i.e., the alternative that acquires strictly more than $\sum_i m_i/2$ votes wins. For simplicity, denote $M_n \equiv \sum_i m_i$. To avoid tie-breaking, we assume that M_n is an odd number.

We assume that the accuracies of voters are independent: let $\{x_i(q_i)\}_{i\in N_n}$ be independent random variables such that

$$x_i(q_i) = \begin{cases} 1 & \text{with probability } q_i\text{, and} \\ 0 & \text{with probability } 1 - q_i. \end{cases} \tag{1}$$

$x_i = 1$ corresponds to the event that a voter i votes for the right alternative, and $x_i = 0$, for the wrong one.[1] Then, the probability that the right alternative is chosen is

$$\Pr\left(\sum_i m_i x_i(q_i) > \frac{M_n}{2}\right). \tag{2}$$

In summary, given strategies $\{q_j\}_{j\in N_n}$, the payoff to each voter i is

$$r_i \Pr\left(\sum_j m_j x_j(q_j) > \frac{M_n}{2}\right) - C_i(q_i). \tag{3}$$

Voters are classified in finite ($K < \infty$) subgroups, $\{N_n^k\}_{k=1,\ldots,K}$. If $i \in N_n^k$, then $(r_i, m_i, C_i) = (r_k, m_k, C_k)$. Denote $\alpha_n^k \equiv |N_n^k|/|N_n|$. Let each subgroup account for positive share: $\alpha_k \equiv \lim_{n\to\infty} \alpha_n^k > 0$ for all k. To make sure some smoothness, we assume that a positive (but arbitrarily small) fraction of voters have a unit vote: $m_k = 1$ for some k.[2]

We focus on symmetric pure Nash equilibria: i.e., an equilibrium is characterized by $\{q_n^k\}_{k=1,\ldots,K}$ such that for all k and for all $i \in N_n^k$,

$$q_n^k = q_i \in \arg\max_{q_i} r_i \Pr\left(\sum_j m_j x_j(q_j) > \frac{M_n}{2}\right) - C_i(q_i). \tag{4}$$

The probability in the equilibrium that the right alternative is chosen is

$$\Pr\left(\sum_k \sum_{i\in N_n^k} m_k x_i(q_n^k) > \frac{M_n}{2}\right). \tag{5}$$

The probability in the equilibrium that the right alternative wins the majority in a subgroup k is

$$\Pr\left(\sum_{i\in N_n^k} x_i(q_n^k) > \frac{|N_n^k|}{2}\right). \tag{6}$$

For the cases of $K = 1$, Martinelli (2004) shows that if $C_1'(1/2) = 0$ and $0 < C_1''(1/2) < \infty$, then (5) converges to $\Phi(d)$ as n grows, where Φ is the cumulative distribution function of $N(0,1)$, and d solves

$$\frac{\phi(d)}{d} = \frac{C_1''(1/2)}{4}, \tag{7}$$

[1] We do not explicitly consider strategic voting here. For discussions on strategic and *naive* voting, see, e.g., Austen-Smith and Banks (1996).

[2] More precisely, it is sufficient for utilizing the local limit theorem of Mcdonald (1979).

where ϕ is the probability density function. Note that for any $d \in (0, \infty)$, there exists $C_1''(1/2) \in (0, \infty)$ that satisfies (7). Moreover, as we can see in Martinelli (2004) (or (19) in Appendix here), the requirement of $C_1'(1/2) = 0$ is equivalent to just requiring that voters do not give up positive investments as long as there remains a possibility of affecting the outcome. Thus, we can interpret any probability of the right choice (not worse than a fair coin toss) in reality as a limit behavior of the model here with no vote weight disparity and an identical cost function that is not so implausible. We find in the next section that such interpretations are also possible with any other disparities.

3 Results and Discussions

For simplicity, denote average values as $A[y] \equiv \sum_k \alpha_k y_k$.

Theorem. *Suppose that $C_k'(1/2) = 0$ and $0 < C_k''(1/2) < \infty$ for all k. Then, as n grows, (5) converges to $\Phi(d)$ where d solves*

$$\frac{\phi(d)}{d} = \frac{1}{4} \frac{A[m^2]}{A[rm^2/C''(1/2)]}. \tag{8}$$

Moreover, for all k, (6) converges to $\Phi(\sqrt{\alpha_k} e_k)$ where e_k solves

$$e_k = \frac{r_k m_k}{C_k''(1/2)} \frac{\sqrt{A[m^2]}}{A[rm^2/C''(1/2)]} d. \tag{9}$$

Thus, any probability of the right choice with any distribution of voters' utilities and vote weights can be justified. Moreover, if we accept the ability indifference (i.e., $C_k = \bar{C}$ for all k), then we can predict the probability with no disparity, and simply by just checking the sign of the covariance of utilities and squared vote weights, figure out whether the performance would deteriorate or not. We summarize these implications below.

Corollary 1. *Consider any $\{(\alpha_k, r_k, \hat{m}_k)\}$ and $P \in (1/2, 1)$. Then, there exists $\bar{C}$ with $\bar{C}'(1/2) = 0$ and $0 < \bar{C}''(1/2) < \infty$ such that: if $\{m_k\} = \{\hat{m}_k\}$ and $C_k = \bar{C}$ for all k, then (5) converges to P. Moreover, for any such $\bar{C}$: if $m_k = 1$ for all k, then (5) converges to $P_0 = \Phi(d_0)$ where d_0 solves*

$$\frac{\phi(d_0)}{d_0} = \frac{A[r\hat{m}^2]}{A[r]A[\hat{m}^2]} \frac{\phi(d)}{d} \tag{10}$$

where d solves $\Phi(d) = P$. (10) implies that $P_0 < P$ if and only if

$$\mathrm{Cov}\left(r, \hat{m}^2\right) > 0. \tag{11}$$

It is sometimes difficult to know about voters' utilities as well as their abilities. However, present behaviors of voters under the existing disparity may provide us some information about them. The following corollary states that learning how often the right alternative wins the majority in each subgroup under the existing disparity, $\{P_k\}$, is enough to predict the performance with no disparity, even though we do not impose the restriction of identical abilities there.

Corollary 2. *Consider any $\{(\alpha_k, \hat{m}_k)\}$ and $\{P_k\}$ with $1/2 < P_k < 1$ for all k. Then, there exists $\{C_k\}$ with $C_k'(1/2) = 0$ and $0 < C_k''(1/2) < \infty$ for all k and $\{r_k\}$ such that: if $\{m_k\} = \{\hat{m}_k\}$, then (6) converges to P_k for all k. Moreover, for any such $\{C_k\}$ and $\{r_k\}$: (i) if $\{m_k\} = \{\hat{m}_k\}$, then (5) converges to $P = \Phi(d)$ where*

$$d = \frac{A[\hat{m}e]}{\sqrt{A[\hat{m}^2]}} \tag{12}$$

where e_k solves

$$P_k = \Phi(\sqrt{\alpha_k} e_k) \tag{13}$$

for all k, and (ii) if $m_k = 1$ for all k, then (5) converges to $P_0 = \Phi(d_0)$ where d_0 solves

$$\frac{\phi(d_0)}{d_0} = \frac{A[\hat{m}e]}{A[e/\hat{m}]A[\hat{m}^2]} \frac{\phi(d)}{d}. \tag{14}$$

(14) implies that $P_0 < P$ if and only if

$$\mathrm{Cov}\left(e/\hat{m}, \hat{m}^2\right) > 0. \tag{15}$$

Finally, we briefly discuss how biased the prediction will be if one ignores the endogeneity.[3] It is equivalent to regarding $\{e_k\}$ as given exogenously: he expects that the probability of the right choice would become $\hat{P}_0 = \Phi(\hat{d}_0)$ under no disparity from $P = \Phi(d)$ with d in (12) under the existing disparity, where

$$\hat{d}_0 = \frac{A[1e]}{\sqrt{A[1^2]}} = A[e]. \tag{16}$$

Thus, he concludes that the performance would not be damaged *if and only if*

$$\hat{d}_0 = A[e] \geq \frac{A[\hat{m}e]}{\sqrt{A[\hat{m}^2]}} = d. \tag{17}$$

[3]There are many literatures on the effects of vote weight with exogenous competences. See, e.g., Nitzan and Paroush (1982).

Then, can it happen that he concludes that the performance would deteriorate by removing existing disparity away though actually it would not? Surely it can. Consider cases with $e_k = a\hat{m}_k^b$ for all k. If $b < 1$, then $\mathrm{Cov}(e/\hat{m}, \hat{m}^2) < 0$, and hence $P_0 > P$. However, for $b = 1$,

$$\frac{A[\hat{m}e]}{\sqrt{A[\hat{m}^2]}} = a\sqrt{A[\hat{m}^2]} > aA[\hat{m}] = A[e]. \tag{18}$$

Thus, $\hat{P}_0 < P < P_0$ everywhere within an interval $b \in (\underline{b}, 1)$ for some $\underline{b} < 1$. In other words, the prediction ignoring the endogeneity is biased in favor of *status quo* vote weight disparity.

Appendix

Proof of Theorem: For simplicity, denote $S_n \equiv \sum_k \sum_{i \in N_n^k} m_k x_i(q_n^k)$ and $S_n^i \equiv S_n - m_k x_i(q_n^k)$ for $i \in N_n^k$. Let $E_n \equiv E[S_n] = \sum_k |N_n^k| m_k q_n^k$ and $V_n \equiv \mathrm{Var}(S_n) = \sum_k |N_n^k| m_k^2 q_n^k (1 - q_n^k)$. Define d_n as

$$d_n \equiv \frac{E_n - M_n/2}{\sqrt{V_n}}.$$

Then, by the central limit theorem,[4] (5) converges to $\Phi(d)$ if $\lim_{n\to\infty} d_n = d$.

The first order conditions of (4) yield[5]

$$r_k \sum_{m=1}^{m_k} \Pr\left(S_n^i = \frac{M_n+1}{2} - m\right) = C_k'(q_n^k). \tag{19}$$

For simplicity, let $P_n^k(m) \equiv \Pr\left(S_n^i = (M_n+1)/2 - m\right)$ for some (and hence for all) $i \in N_n^k$. Then, rearranging (19) yields

$$\frac{\sqrt{V_n}\sum_{m=1}^{m_k} P_n^k(m)/m_k}{\phi(d_n)} \frac{\phi(d_n)}{d_n} = \frac{C_k'(q_n^k)/(r_k m_k)}{E_n/M_n - 1/2} \frac{V_n}{M_n}. \tag{20}$$

First, we show that $\limsup_{n\to\infty} d_n < \infty$. Suppose, on the contrary, that along some subsequence, $\lim_{n\to\infty} d_n = \infty$. If $\lim_{n\to\infty} d_n/|N_n|^{1/2} > 0$, then since $\lim_{n\to\infty} \sqrt{V_n}/d_n < \infty$, the LHS of (20) converges to 0. Otherwise, note that by (19), $\max_k \left|q_n^k - 1/2\right| \to 0$. Thus, $V_n/|N_n| \to \sum_k \alpha_k m_k^2/4 > 0$. Therefore, since $m_k = 1$ for some k, by the local limit theorem,[6]

$$\lim_{n\to\infty} \frac{\sqrt{V_n} P_n^k(m)}{\phi(d_n)} = 1 \text{ for all } k, \text{ for } 1 \le m \le m_k. \tag{21}$$

Thus, the first term of the LHS converges to 1, and hence the LHS to 0 again. However, let $k(n) \in \arg\max_k q_n^k$. Then,

$$\frac{C_{k(n)}'(q_n^{k(n)})}{E_n/M_n - 1/2} \ge \frac{C_{k(n)}'(q_n^{k(n)})}{q_n^{k(n)} - 1/2} \ge \min_k \frac{C_k'(q_n^k)}{q_n^k - 1/2} \to \min_k C_k''(1/2) > 0 \text{ as } n \to \infty.$$

Thus, contradictorily, the RHS for $k(n)$ does not converge to 0.

Therefore, $\limsup_{n\to\infty} d_n < \infty$ holds. Then, by the local limit theorem as above, (21) holds again. Thus, for all k, k',

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{m_k} P_n^k(m)/m_k}{\sum_{m'=1}^{m_{k'}} P_n^{k'}(m')/m_{k'}} = 1,$$

[4] See, e.g., Feller (1971).
[5] Note that a voter can affect the "outcome" only when his vote is pivotal.
[6] See, e.g., McDonald (1979).

which implies, by (19),

$$\lim_{n\to\infty} \frac{C_k'(q_n^k)/(r_k m_k)}{C_{k'}'(q_n^{k'})/(r_{k'} m_{k'})} = 1. \tag{22}$$

Therefore,

$$\begin{aligned} \frac{C_k'(q_n^k)/(r_k m_k)}{E_n/M_n - 1/2} &= \frac{1}{\frac{\sum_{k'} \alpha_n^{k'} r_{k'} m_{k'}^2}{\sum_{k''} \alpha_n^{k''} m_{k''}} \frac{C_{k'}'(q_n^{k'}))/r_{k'} m_{k'}}{C_k'(q_n^k)/(r_k m_k)} \frac{q_n^{k'}-1/2}{C_{k'}'(q_n^{k'})}} \\ &\to \frac{\sum_{k'} \alpha_{k'} m_{k'}}{\sum_k \alpha_k r_k m_k^2 / C_k''(1/2)} \text{ as } n \to \infty. \end{aligned} \tag{23}$$

Since $\lim_{n\to\infty} V_n/M_n = (1/4)\left(\left(\sum_k \alpha_k m_k^2\right) / \left(\sum_{k'} \alpha_{k'} m_{k'}\right)\right)$, (20), (21) and (23) imply that d must solve (8).

Define d_n^k as

$$d_n^k \equiv \frac{|N_n^k|(q_n^k - 1/2)}{\sqrt{|N_n^k| q_n^k (1-q_n^k)}}.$$

Then, since

$$d_n = \sum_k m_k d_n^k \sqrt{\frac{|N_n^k| q_n^k (1-q_n^k)}{\sum_{k'} |N_n^{k'}| m_{k'}^2 q_n^{k'} (1-q_n^{k'})}}, \tag{24}$$

(22) and the central limit theorem again imply (9). □

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