Two-Stage-Partitional Representation and Dynamic Consistency[1]

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Abstract

In this paper, we study an individual who faces a three-period decision problem when she accepts the partitional signal twice at the different times. Adding a new axiom to a Third-Order Belief Representation in Takeoka (2007), we characterize a Two-Stage-Partitional Representation, which make the property of signals more flexible. And, under the context of a Two-Stage-Partitional Representation, we show the subjective version of Dynamic Consistency among two different preference relations over menus.

Keywords: Subjective State Space, Preference for Flexibility, Partitional Signal, Two-Stage-Partitional Representation, Dynamic Consistency

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1 Introduction

1.1 Background

In our dairy decisions, probabilities are not given while in von Neumann-Morgenstern Expected Utility Theorem, probabilities are given. This implies assuming that probabilities are taken as primitives is restrictive. Instead, it is realistic for a Decision Maker (henceforth DM) not to choose her probability but to choose her action.

Since the consequence of her action is uncertain, we consider a state space to model the uncertainty. Subjective probability models assuming that a state space is taken as a primitive are introduced by Savege (1954) and Anscombe-Aumann (1963). In their models, however, their assumptions are restrictive because assuming that a state space is taken as a primitive means that an observer knows all the uncertainties she faces. Therefore, a state space should not be taken as a primitive and it should be derived in the model.

The derivation of a subjective state space is studied by Kreps (1979). He considers preference over menus of alternatives to derive the subjective state space. If the DM knows that she has the uncertainties regarding her future preference over alternatives, her ranking of menus reflects how she perceives the future uncertainties. From the ranking of menus, Kreps derives the set of future preferences - the set of possible ex post preferences, which is called the subjective state space.

However, the subjective state space is not decided uniquely in Kreps. This problem is solved in Dekel, Lipman and Rustichini (2001, henceforth DLR). DLR refines Kreps by enriching the choice object to menus of lotteries over alternatives. In both Kreps and DLR, an observer can identify all the uncertainties she perceives by the derivation of a subjective state space but an observer can't identify how the DM expects uncertainties to be resolved as time goes on.

Takeoka (2007) derives the subjective state space and the filtration in a dynamic setting without taking them as primitives by considering preference over menus of menus of Anscombe-Aumann acts. Moreover, Takeoka identifies the subjective probability measure on the subjective state space which is not identified even in DLR because the state-independence of risk preference is applied in Takeoka, but the state-dependence of risk preference is applied in DLR.

On the other hand, under the menu setting, Dekel, Lipman, Rustichini and Sarver (2007) and Dekel et al. (2009) develop DLR. Also Kochov (2007), Ergin and Sarver (2010), Higashi and Hyogo (2012) study the extension of DLR by weakening DLR's assumptions of Completeness, Independence and Continuity, respectively. Furthermore, Ahn and Sarver (2013) discusses the unforeseen contingencies. In the dynamic setting, Krishna and Sadowski (2014) shows Dynamic Preference for Flexibility Representation by introducing a Markov Process on states of the world.

Our paper is located on an extension of these papers which deal with the subjective state space.

1.2 Motivation and Goal

Riella (2013) studies the appropriate version of Dynamic Consistency when the state space is subjective. It's natural for rational agents to follow the property of Dynamic Consistency when the state space is exogenous, while it's not easy to find whether it satisfies the property of Dynamic Consistency when the state space is endogenous because an observer doesn't know the state space the DM faces. In Riella, the domain is the same as DLR, which is menus of lotteries over alternatives, and a different setting from DLR is that an objective state signal is sent to a DM before choosing a menu.

In this paper, we pursue the subjective version of Dynamic Consistency using the framework of Takeoka, which is a three-period decision problem that the DM accepts the belief signal twice at the different times. In order to achieve our goal, we show a Two-Stage-Partitional Representation different from Takeoka, in which the subjective signals are based on her belief so we can't apply Riella's method directly. Therefore, we change the subjective belief signals to the subjective partitional signals to build a new representation. It enables us to consider the subjective version of Dynamic Consistency. Riella shows the subjective version of Dynamic Consistency by using the Flexibility Consistency when the subjective state is finite and a state signal the DM faces is specific. On the other hand, we show[2] the subjective version of Dynamic Consistency when the subjective state is finite and each signal the DM faces is partition. Moreover, Riella focus on lotteries but ours on acts.

Our contributions are the followings. First of all, adding a new axiom to a Third-Order Belief Representation in Takeoka, we characterize a Two-Stage-Partitional Representation, which is more flexible (I explain it in section 3.4). Next, we show the subjective version of Dynamic Consistency among two different preference relations over menus using a Two-Stage-Partitional Representation.

1.3 Related Literature and Outline

As related literatures, we should explain Dillenberger, Lleras, Sadowski and Takeoka (2014, henceforth DLST) and Dillenberger and Sadowski (2014, henceforth DS).

In DLST, they work in a setup of menus of Anscombe-Aumann acts and send a subjective signal to a DM at the point between choosing a menu and choosing an act. As a result, they represent Subjective Learning Representation and Partition Learning Representation. Also in DLST, the information structure is identified uniquely. In DS, they state Generalized-Partition Representation using DLST framework and consider the subjective version of Dynamic Consistency. DS is probably the closest paper to mine. However, compared to our model, the domain and the number of accepting signals are different. In this sense, our paper is absolutely different from two papers above.

The remaining of the paper is organized as follows. In section 2, we summarize Takeoka and set some axioms required later. In section 3, we add a new axiom and state the theorem of a Two-Stage-Partitional Representation. In section 4, we consider the subjective version of Dynamic Consistency

[2] We have to put a strong assumption. I explain it in section 4

under the context of a Two-Stage-Partitional Representation. In section 5, we state the conclusion and discuss the future work.

2 Preliminaries

As an extension of the previous literature mentioned above, our model is based on Takeoka. In our model, the domain, primitives and periods of decisions are almost all the same as Takeoka but the property of signals is different, which leads to totally different results. In detail, different from Takeoka which uses the subjective belief signal, we use the subjective partitional signal in our model. In this section, we provide an overview of Takeoka.[3]

2.1 Domain and Timing

Let Ω be a finite objective state space (a DM may have some subjective states other than Ω). Also let Z be a non-empty finite set of outcomes and $\Delta(Z)$ be the set of all Borel probability measures over Z, which is a compact metric space under the weak metric convergence topology. Furthermore, he calls h: $\Omega \to \Delta(Z)$ an act and $\mathbb{H} = \{h \mid \Omega \to \Delta(Z)\}$ the set of all acts, which is a compact metric space under the product topology.

Let $K(\mathbb{H})$ be the set of all non-empty compact subsets of $\mathbb{H}$. He calls $K(\mathbb{H})$ with Hausdorff metric menus of acts. Also let $K(K(\mathbb{H}))$ be the set of all non-empty compact subsets of $K(\mathbb{H})$ with Hausdorff metric. He also calls $K(K(\mathbb{H}))$ menus of menus of acts. Consider preference $\succsim$ over $K(K(\mathbb{H}))$.

The DM has in mind the following timing of decisions. In period 0, the DM chooses a menu of menus of acts $x_0 \in K(K(\mathbb{H}))$; in period 1^-, she receives a subjective belief signal S^1; in period 1, she chooses a menu of acts $x_1 \in x_0$; in period 2^-, she receives a subjective belief signal S^2; in period 2, she chooses an act $h \in x_1$; in period 2^+, a state is realized and she receives the lottery. We focus on the dynamic situation in three stage decision making.

2.2 Axioms

We turn to the axiomatic foundation of the model. Takeoka considers the following axioms for a binary relation $\succsim$.

Axiom 1 (Order). $\succsim$ is complete and transitive.

Axiom 2 (Continuity). For all $x_0 \in K(K(\mathbb{H}))$,

[3] Takeoka defines a decision tree, which is a pair consisting of a state space and a filtration. Depending on the DM's exhibiting Preference for Flexibility, we can have a variety of subjective decision trees. However, in our paper, we focus on the DM who exhibits Preference for Flexibility in every periods.

$$\{z_0 \in K(K(\mathbb{H}))| \, x_0 \gtrsim z_0\} \text{ and } \{z_0 \in K(K(\mathbb{H}))| \, z_0 \gtrsim x_0\} \text{ are closed}$$

Axiom 3 (Strong Non-degeneracy). There exists $l, l' \in \Delta(Z)$ such that $\{\{l\}\} \succ \{\{l'\}\}$

Axiom 4 (Independence). For all $x_0, y_0, z_0 \in K(K(\mathbb{H}))$ and for all $\lambda \in (0, 1]$

$$x_0 \succ y_0 \Rightarrow \lambda x_0 + (1-\lambda) z_0 \succ \lambda y_0 + (1-\lambda) z_0$$

Axiom 5 (Monotonicity). For all $x_0, x'_0 \in K(K(\mathbb{H}))$,

$$x_0 \subset x'_0 \Rightarrow x'_0 \gtrsim x_0$$

Axiom 6 (Aversion to Commitment). For all $x'_0 \in K(K(\mathbb{H}))$ and for all finite $x_0 \in K(K(\mathbb{H}))$,

$$x'_0 \cup \{\cup_{x_1 \in x_0} x_1\} \gtrsim x'_0 \cup x_0$$

Axiom 7 (Risk Preference Certainty).

$$\text{For all } x_0 \in K(K(\mathbb{H})), \; x_0 \sim o(x_0)$$

For any $h \in \mathbb{H}$, $o_1(h) \stackrel{\text{def}}{=} \{h' \in \mathbb{H} | \{\{h(\omega)\}\} \gtrsim \{\{h'(\omega)\}\} \text{ for all } \omega\}$, for each $x_1 \in K(\mathbb{H})$, $o_1(x_1) \stackrel{\text{def}}{=} \cup_{h \in x_1} o_1(\mathrm{h})$, and for each $x_0 \in K(K(\mathbb{H}))$, $o(x_0) \stackrel{\text{def}}{=} \{o_1(x_1) \mid x_1 \in x_0\}$.

The first five axioms are standard. The sixth axiom, Aversion to Commitment, means that the DM weakly wants to delay her decision. And the seventh axiom, Risk Preference Certainty, means that she totally knows the ranking over lotteries.

2.3 Representation

Take any third-order belief $\mu_0 \in \Delta(\Delta(\Delta(\Omega)))$ and non-constant continuous mixture linear function $u : \Delta(Z) \to \mathbb{R}$. Define the functional form U_0: $K(K(\mathbb{H})) \to \mathbb{R}$.

$$U_0(x_0) = \int_{\Delta(\Delta(\Omega))} \max_{x_1 \in x_0} U_1(x_1, \mu) \, \mathrm{d}\mu_0(\mu) \quad \text{where}$$

$$U_1(x_1, \mu) = \int_{\Delta(\Omega)} \max_{h \in x_1} U_2(h, p) \, \mathrm{d}\mu(p) \quad \text{for } \mu \in \Delta(\Delta(\Omega)) \quad \text{and}$$

$$U_2(h, p) = \sum_{\omega \in \Omega} u(h(\omega)) p(\omega) \quad \text{for } p \in \Delta(\Omega)$$

$\mu_0 \in \Delta(\Delta(\Delta(\Omega)))$ means her belief about the first signal (S^1) so that $\mu \in \Delta(\Delta(\Omega))$ is regarded as the first signal (S^1). Also the history of signals (S^1, S^2) in the time line is denoted by $p \in \Delta(\Omega)$. We should pay attention to applicants before the first signal (S^1) denoted by $\mu_1 \in \Delta(\Delta(\Omega))$, and the belief on Ω just after the first signal (S^1) denoted by $\pi \in \Delta(\Omega)$ in order to smoothly progress our discussion in section 3 although they're not written above.

Definition 1. Preference $\succsim$ on $K(K(\mathbb{H}))$ admits a Third-Order Belief Representation if there exists a functional form above that represents $\succsim$.

Theorem 1.[4] Preference $\succsim$ satisfies Axioms 1–7 if and only if it admits a Third-Order Belief Representation.

3 Model

In this section, we provide the model setting and characterize a Two-Stage-Partitional Representation. As mentioned above, the domain and primitives are quite similar to Takeoka but the timing of decisions are different, which leads to totally different representation.

3.1 Setup

The DM has in mind the following timing of decisions:

In period 0, she chooses a menu of menus of acts $x_0 \in K(K(\mathbb{H}))$.

In period 1^-, she receives the first partition on Ω: ρ_1.

In period 1, she chooses a menu of acts $x_1 \in x_0$.

In period 2^-, she receives the second partition on Ω: ρ_2.

In period 2, she chooses an act $h \in x_1$.

In period 2^+, a state is realized and she receives the lottery.

The only change towards Takeoka is the property of signals. Takeoka uses the subjective belief signal such as S^1 and S^2. In our model, however, we use the subjective partitional signal say, ρ_1 and ρ_2.

3.2 Axioms

We turn to the axiomatic foundation in our model. Takeoka considers the seven axioms above for a binary relation $\succsim$ and we add a new axiom to Takeoka in order to characterize a functional form in the timing of decisions above. Before explaining a new axiom, we should introduce the concept of a composite act.

Definition 2. A composite act fIg is defined as follows. For any event $I \in 2^\Omega$, and acts $f, g \in \mathbb{H}$,

[4] In addition, Takeoka states on the uniqueness of the representation (μ_0, u) and it also pins down the uniqueness of probability measure.

$$fIg(\omega) = \begin{bmatrix} f(\omega) & \text{if } \omega \in I \\ g(\omega) & \text{if } \omega \notin I \end{bmatrix}$$

Axiom 8[5] (Indifference to State Contingent Commitment on Menus of Menus of Acts)

For all $f, g \in \mathbb{H}$, there exists $I \in 2^{\Omega}$, such that $\{\{fIg\}\} \sim \{\{f, g\}\}$

Since $\{\{fIg\}\}$ means a commitment menu of menus of acts and $\{\{f, g\}\}$ is the most flexible menu of menus of acts, this axiom means that the DM is indifferent between committing to the composite act, which she decides ex ante, and choosing one of two acts, which she decides ex post.

3.3 Two-Stage-Partitional Representation

Take any probability measure θ_1 on $\Delta(\Omega)$ and non-constant continuous mixture linear function u : $\Delta(Z) \rightarrow \mathbb{R}$. Consider the functional form V_0: $K(K(\mathbb{H})) \rightarrow \mathbb{R}$.

$$V_0(x_0) = \sum_{I_1 \in \rho_1} \max_{x_1 \in x_0} V_1(x_1, I_1)\theta_1(I_1) \quad \text{where}$$

$$V_1(x_1, I_1) = \sum_{I_2 \in \rho_2} \max_{h \in x_1} V_2(h, I_2)\, \theta_1(I_2)$$

$$V_2(h, I_2) = \sum_{\omega \in I_2} u(h(\omega))\, \theta_1(\omega \mid I_2)$$

$\theta_1 \in \Delta(\Omega)$ is an initial belief on Ω and ρ_1 is the first subjective partitional signal and I_1 are partitions which include each state space divided Ω into ρ_1. Similarly, ρ_2 is the second subjective partitional signal given the first partitional signal and I_2 are partitions which include each state space divided Ω into ρ_2.

Definition 3. Preference $\succsim$ on $K(K(\mathbb{H}))$ admits a Two-Stage-Partitional Representation if there exists a functional form above that represents $\succsim$.

Theorem 2. Preference $\succsim$ satisfies Axioms 1–8 if and only if it admits a Two-Stage-Partitional Representation.

Proof of Theorem 2

An idea in DLST can be applied here to prove Theorem 2. Necessity of the axioms is obvious. Therefore, we show sufficiency.

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[5] As a development of the axiom used in DLST, this axiom is mentioned.

Proof.

What to be shown is that "signals in Third-Order Belief Representation are partition." ⇔ "$\sigma(\pi) \cap \sigma(\pi') = \emptyset$ for all π with $\pi \neq \pi'$."[6] The right direction part is obvious. Then we show the left direction part.

Take U_0 a Third-Order Belief Representation. Now, assume that there exist π and π' such that $\pi \neq \pi'$ and $\sigma(\pi) \cap \sigma(\pi') \neq \emptyset$. In addition, $\omega^* \in \sigma(\pi) \cap \sigma(\pi')$. Then, there exist acts f and g such that $\sum_\omega f(\omega)\pi(\omega) > \sum_\omega g(\omega)\pi(\omega)$ and $\sum_\omega f(\omega)\pi'(\omega) < \sum_\omega g(\omega)\pi'(\omega)$. Take $\varepsilon > 0$ sufficiently small. Define $g^\varepsilon(\omega)$ as follows:

$$g^\varepsilon(\omega) = \begin{cases} g(\omega^*) + \varepsilon & \text{if } \omega = \omega^* \\ g(\omega) & \text{if } \omega \neq \omega^* \end{cases}$$

Consider a menu of menus of acts $\{\{f, g^\varepsilon\}\}$. It is possible to choose g^ε from $\{\{f, g^\varepsilon\}\}$ because $\sum_\omega f(\omega)\pi'(\omega) < \sum_\omega g(\omega)\pi'(\omega)$.

Therefore, $U_0(\{\{f, g^\varepsilon\}\}) \neq U_0(\{\{f, g\}\})$ and $U_0(\{\{f, g^\varepsilon\}\}) \rightarrow U_0(\{\{f, g\}\})$ as $\varepsilon \rightarrow 0$.

What's important is that $f I g^\varepsilon(\omega^*) \neq f(\omega^*)$ when $\{\{f I g^\varepsilon\}\} \sim \{\{f, g^\varepsilon\}\}$ holds.

Fix ε arbitrary and take $\delta > 0$ sufficiently small. Define $f^\delta(\omega)$ as follows:

$$f^\delta(\omega) = \begin{cases} f(\omega^*) + \delta & \text{if } \omega = \omega^* \\ f(\omega) & \text{if } \omega \neq \omega^* \end{cases}$$

Similarly, consider a menu of menus of acts $\{\{f^\delta, g^\varepsilon\}\}$. It is possible to choose f^δ from $\{\{f^\delta, g^\varepsilon\}\}$ because $\sum_\omega f^\delta(\omega)\pi(\omega) > \sum_\omega g^\varepsilon(\omega)\pi(\omega)$.

Therefore, $U_0(\{\{f^\delta, g^\varepsilon\}\}) \neq U_0(\{\{f, g^\varepsilon\}\})$ and $U_0(\{\{f^\delta, g^\varepsilon\}\}) \rightarrow U_0(\{\{f, g^\varepsilon\}\})$ as $\delta \rightarrow 0$.

Also there exists I' such that $\{\{f^\delta I' g^\varepsilon\}\} \sim \{\{f^\delta, g^\varepsilon\}\}$. As a result,

$$f^\delta I' g^\varepsilon(\omega^*) = f^\delta(\omega^*) = f(\omega^*) + \delta.$$

As the sequence $\{\{f^\delta I' g^\varepsilon\}\}_{\delta \rightarrow 0}$ is in the compact set $K(K(\mathbb{H}))$, it converges to a point h in the compact set. The discussion above leads to the following:

$$h(\omega^*) = f(\omega^*) + 0 = f(\omega^*)$$

and

$$h(\omega) \in \{\{f(\omega), g^\varepsilon(\omega)\}\}$$

Thus, under an event $\hat{I} \subset \Omega$,

$$h = f \hat{I} g^\varepsilon$$

Therefore,

$$\{\{f \hat{I} g^\varepsilon\}\} = \{\{h\}\} \sim \{\{f, g^\varepsilon\}\}$$

These results contradict with the idea that $f I g^\varepsilon(\omega^*) \neq f(\omega^*)$ when $\{\{f I g^\varepsilon\}\} \sim \{\{f, g^\varepsilon\}\}$ holds. ■

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[6] σ is a support of π, that is, $\sigma(\pi) \subset \Omega$ and $\pi \in \Delta(\Omega)$ is defined as the belief on Ω just after the first signal $\mu \in \Delta(\Delta(\Omega))$.

A Two-Stage-Partitional Representation is better to a Third-Order Belief Representation in two ways. First, a Two-Stage-Partitional Representation captures a reality in the world much more than a Third-Order Belief Representation. Next, we can consider the subjective version of Dynamic Consistency using a Two-Stage-Partitional Representation. We discuss the first point with examples in the next, while we consider the second point in section 4.

3.4 Examples of two Representations

The difference of the property between the belief signals and the partitional signals is caused by different sources of information. Let's consider the DM who will buy an asset in a three-period decision game. She chooses a menu of menus today, and she will accept the first subjective signal and chose a menu tomorrow, and she will accept the second subjective signal and chose an act the day after tomorrow.

At first, we ponder over a Third-Order Belief Representation. Note that the information she accepts twice at the different timings have to be correlated with each other, so we can think about the first and second belief signals as a series of information which is dependent on each other over periods. For example, we interpret the first belief signal as an expected value of GDP, and the second belief signal as a definitive value of GDP.

On the other hand, in terms of the partitional signals, it's not always the case that the information she accepts twice at the different timings are correlated with one another, so it is sometimes possible that we regard the first and second partitional signals as the information which are independent on one another over periods[7]. For example, we interpret the first partitional signal as an exchange rate, and the second partitional signal as a defective value of GDP.

4 Dynamic Consistency

Riella shows that Flexibility Consistency is equivalent to the subjective version of Dynamic Consistency when the subjective state is finite and an objective state signal the DM faces is specific. On the other hand, we consider building the proper version of Dynamic Consistency when the subjective state is finite and each signal the DM faces is partition. In my setting, we have to put an assumption that an observer can observe partitional signals.[8]Also, in terms of updating, even if some states are unforeseen, which means she is aware of new states in the world over time, it can be interpreted as a reverse Bayesian updating when the role of preference relation before signals and that

[7] Everything is all right even if the information she accepts twice at the different timings are correlated with one another.

[8] This is a strong assumption because subjective signals are not identified in this paper.

after signals are reversed.[9]

4.1 Setup

We work within the setup in our model shown in the section 3. Again, let us put an assumption that an observer can observe partitional signals, which means he can observe the each partitional signal. However, each partitional signal is interpreted by a DM, so an observer cannot realize how she interprets each signal.

Definition 4. The following statements are defined.

1. Consider preference $\succsim$ over $K(K(\mathbb{H}))$. Take any θ_1,

$$V_0(\mathrm{x}_0) = \sum_{I_1 \in \rho_1} \max_{x_1 \in x_0} V_1(x_1, I_1)\theta_1(I_1)$$

$$\begin{pmatrix} V_1(x_1, I_1) = \sum_{I_2 \in \rho_2} \max_{\mathrm{h} \in x_1} V_2(h, I_2)\, \theta_1(I_2) \\ V_2(h, I_2) = \sum_{\omega \in I_2} u(h(\omega))\, \theta_1(\omega \mid I_2) \end{pmatrix}$$

2. Consider preference $\succsim^*$ over $K(\mathbb{H})$. For any two menus of acts α and β in the menu of menus x_0,

$$\alpha \succsim^* \beta \Leftrightarrow \sum_{I_2 \in \rho_2} \max_{\mathrm{h} \in \alpha} V_2(h, I_2)\, \theta_1(I_2) \geq \sum_{I_2 \in \rho_2} \max_{\mathrm{h} \in \beta} V_2(h, I_2)\, \theta_1(I_2)$$

$$\left(V_2(h, I_2) = \sum_{\omega \in I_2} u(h(\omega))\, \theta_1(\omega \mid I_2) \right)$$

3. *Supp* (θ_1)[10] $= \Omega$

We will use $\succsim^*$ over menus to represent her preference just before the first partitional signal and $\{\succsim^*_{\omega_i}\}_{\omega_i \in I_{1k}}$ (k = 1, 2, ••, n[11]) over menus to represent her preference just after the first partitional signal. Condition 2 in the definition says she chooses a menu from the selected menu of menus knowing that the partitional signal will come twice later in order to maximize her *ex ante* expected utility, taking into account that she will choose the best act in the future. Condition 3 in the definition asserts Ω contains no redundant states.

4.2 Dynamic Consistency between Preferences over Menus

We consider Dynamic Consistency between preferences over menus for the following steps. In the first step, we argue an objective state signal case. In the second step, we consider the partitional signal case. That is, we first argue $\succsim^*$ and $\succsim^*_{\omega_i}$ for each $\omega_i \in \Omega$, which means a preference

[9] A reverse Bayesianism rule is first introduced in Karni and Viero (2013).
[10] The support of θ_1 for a given probability measure θ_1. It means that every state in Ω has positive probability.
[11] n is less than the number of states.

relation when realizing the specific state $\omega_i \in \Omega$. And later we consider $\succsim^*$ and $\{\succsim^*{}_{\omega_i}\}_{\omega_i \in I_{1k}}$. As noticed above, there are two changes related to Riella: (1) acts rather than lotteries; and (2) the partitional signal rather than the state signal. Define the Preference for Flexibility, which is due to DLR.

Definition 5. A binary relation $\succsim^*$ over $K(\mathbb{H})$ values flexibility more than $\succsim^*{}_{\omega_i}$ over $K(\mathbb{H})$ if, for any two menus α and β in the menu of menus x_0 with $\beta \subseteq \alpha$.

$$\alpha \succ^*{}_{\omega_i} \beta \Rightarrow \alpha \succ^* \beta$$

It is also well-known by DLR that $\succsim^*$ values more than flexibility than $\succsim^*{}_{\omega_i}$ if and only if the subjective state space that represents $\succsim^*$ is larger than the subjective state space that represents $\succsim^*{}_{\omega_i}$. The following definition is introduced by Riella, which is equivalent to the subjective version of Dynamic Consistency in Riella.

Definition 6. Flexibility Consistency

For any menu $\alpha \in K(\mathbb{H})$ and menu $\beta \in K(\widehat{\mathbb{H}})$[12] in the menu of menus x_0, $\alpha \succ^*{}_{\omega_i} \beta$ and $\beta \succsim^* \alpha$ implies that there exists a menu γ such that $\alpha \cup \beta \cup \gamma \backsim^*{}_{\omega_i} \alpha \cup \gamma$, but $\alpha \cup \beta \cup \gamma \succ^* \alpha \cup \gamma$

Lemma 1. If $\succsim^*$ and $\succsim^*{}_{\omega_i}$ satisfy Flexibility Consistency, then $\succsim^*$ values flexibility more than $\succsim^*{}_{\omega_i}$.

It means that any disagreement between $\succsim^*$ and $\succsim^*{}_{\omega_i}$ is only a desire for flexibility. $\succsim^*$ values more than $\succsim^*{}_{\omega_i}$. Note that we require the existence of γ and $\beta \in K(\widehat{\mathbb{H}})$ due to a technical reason in order to complete Theorem 3. The next lemma was shown by Riella. The following theorem is the main result in the first step we state above.

Theorem 3. The following statements are equivalent.

1. $\succsim^*$ and $\succsim^*{}_{\omega_i}$ satisfy Flexibility Consistency

2. Let Ω and ω_i be the unique subjective state space of $\succsim^*$ and $\succsim^*{}_{\omega_i}$, respectively. For any two menus α and β in the menu of menus x_0 with

$$\max_{h \in \alpha} V_2(h, I_2) = \max_{h \in \beta} V_2(h, I_2) \text{ for all } \omega \in \Omega \setminus \omega_i,$$

$$\alpha \succsim^* \beta \Leftrightarrow \alpha \succsim^*{}_{\omega_i} \beta$$

Proof of Theorem 3.

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[12] We define $\widehat{\mathbb{H}} = \{h \mid \Omega \rightarrow \text{int}\,\Delta(Z)\}$ and $K(\widehat{\mathbb{H}})$ is the set of all subsets of $\widehat{\mathbb{H}}$.

The way of this proof is almost all the same as the proof of main theorem in Riella. ■

From now on, we consider the partitional signal case. Riella also states that multiple signals can be interpreted as the partitional signal if the condition below holds.

Definition 7.
For every subset J of I_{1k}, there exists menus α and β in the menu of menus x_0 with $\beta \subseteq \alpha$, such that $\alpha \backsim^*_{\omega_i} \beta$ for all $\omega_i \in J$, but $\alpha \succ^* \beta$.

Definition 8.
For any two menus α and β in the menu of menus x_0, if $\alpha \succsim^*_{\omega_i} \beta$ for every $\omega_i \in I_{1k}$, then $\alpha \succsim^* \beta$.

Definition 7 means any subsets of the relation in $\{\succsim^*_{\omega_i}\}_{\omega_i \in I_{1k}}$ do not waste of all the flexibility represented by $\succsim^*$.

Proposition 1. $\succsim^*$ and $\succsim^*_{\omega_i}$ satisfy Flexibility Consistency. $\{\succsim^*_{\omega_i}\}_{\omega_i \in I_{1k}}$ (k = 1, 2, ••, n) and $\succsim^*$ satisfy Definition 7 and 8 if and only if the collection I_{1k} (k = 1, 2, ••, n) used in $\{\succsim^*_{\omega_i}\}_{\omega_i \in I_{1k}}$ is a partition of Ω.

Proof of Proposition 1.
The way of this proof is almost all the same as the proof of Proposition in Riella. ■

The following result is our main theorem in section 4. Part2 in the theorem can be interpreted the appropriate version of Dynamic Consistency when the subjective state is finite and each signal the DM faces is partition.

Theorem 4. The following statements are equivalent.

1. $\succsim^*$ and $\succsim^*_{\omega_i}$ satisfy Flexibility Consistency. $\{\succsim^*_{\omega_i}\}_{\omega_i \in I_{1k}}$ (k = 1, 2, ••, n) and $\succsim^*$ satisfy Definition 7 and 8.

2. Let Ω and I_{1k} be the unique subjective state space of $\succsim^*$ and $\{\succsim^*_{\omega_i}\}_{\omega_i \in I_{1k}}$, respectively. For any two menus α and β in the menu of menus x_0 with
$$\max_{h \in \alpha} V_2(h, I_2) = \max_{h \in \beta} V_2(h, I_2) \text{ for all } \omega \in \Omega \setminus I_{1k},$$
$$\alpha \succsim^* \beta \Leftrightarrow \alpha \succsim^*_{I_{1k}} \beta$$

3. After the partitional signal, in terms of the belief of the state space, the Bayesian updating is applied.

Proof of Theorem 4.

This part has been revised.

5 Conclusion and Discussion

In this paper, we show a Two-Stage-Partitional Representation and build the subjective version of Dynamic Consistency among two different preference relations over menus before and after the first signal when the subjective state is finite and each signal the DM faces is partition. Theorem 2 and Theorem 4 are our main theorems.

However, changing the subjective partitional signal to the objective partitional signal in section 4 is too restrictive. Therefore, as a future issue, we should consider the elicitation of both the first and the second subjective partitional signal in the context of a Two-Stage-Partitional Representation so that we will consider the subjective version of Dynamic Consistency without such a restrictive condition.

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